A QUANTITATIVE SHRINKING TARGET RESULT ON STURMIAN SEQUENCES FOR ROTATIONS

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Abstract. Let R_α be an irrational rotation of the circle, and code the orbit of any point x by whether $R^i_\alpha(x)$ belongs to $[0, \alpha)$ or $[\alpha, 1)$ – this produces a Sturmian sequence. A point is undetermined at step j if its coding up to time j does not determine its coding at time $j + 1$. We prove a pair of results on the asymptotic frequency of a point being undetermined, for full measure sets of α and x.

1. Introduction

1.1. **Statement of the problem and main results.** In this paper we study a shrinking target problem. Let $\alpha \in [0, 1)$, let $R_\alpha : [0, 1) \to [0, 1)$ be the rotation $R_\alpha(x) = x+\alpha \pmod 1$, and let λ denote Lebesgue measure. The following theorem, due to Weyl, is well known.

Theorem 1.1. *Let $\alpha \notin \mathbb{Q}$. Then for any $x, y \in [0, 1)$, and any $\epsilon > 0$,*

$$\lim_{N\to\infty} \frac{\sum_{i=1}^N \chi_{B(y,\epsilon)}(R^i_\alpha x)}{\sum_{i=1}^N \lambda(B(y, \epsilon))} = 1.$$

That is, the asymptotic for the number of visits of the orbit of x to the target set $B(y, \epsilon)$ by step N is given by the sum of the size of the target over those N steps.

The statement is written here in a slightly unusual way – the denominator is clearly $N2\epsilon$ (identifying $[0, 1)$ with S^1). But it suggests the following sort of problem. Let $\{B_i\}$ be a sequence of measurable sets in $[0, 1)$. What can be said about the behavior of $\sum_{i=1}^N \chi_{B_i}(R^i_\alpha x)$; in particular, is it asymptotic to $\sum_{i=1}^N \lambda(B_i)$?

This is, of course, an enormously varied problem. Cases which have generated significant interest are *shrinking target problems*, in which the B_i form a decreasing, nested chain. By the Borel-Cantelli Lemma, the cases of real interest are when $\sum_{i=1}^\infty \lambda(B_i) = \infty$. Several results on this problem for rotations and for interval exchange transformations are contained in [CC17], including the following.

Theorem 1.2. [CC17] *For all α satisfying an explicit, full measure diophantine condition, and for any sequence $\{r_i\}$ such that ir_i is non-increasing and $\sum_{i=1}^\infty r_i = \infty$, and any y*

$$\lim_{N\to\infty} \frac{\sum_{i=1}^N \chi_{B(y,r_i)}(R^i_\alpha x)}{\sum_{i=1}^N 2r_i} = 1$$

for almost every x.

In this paper we consider another shrinking target problem for rotations, but one whose targets arise in a very different way. Rather than being subject to some pre-determined analytic constraint (as for the sequence $\{r_i\}$) the targets arise from the dynamics of the rotation itself.

Let α be given. Let $\mathcal{P} = \{A_0, A_1\}$ be the partition of $[0,1)$ given by $A_0 = [0,\alpha), A_1 = [\alpha, 1)$. The bi-infinite sequences $(c_i(x))_{i\in\mathbb{Z}}$ defined by $c_i(x) = j$ if $R^i_\alpha x \in A_j$ are known as Sturmian sequences (see, e.g. [BFMS02], Ch. 6). These are sequences of minimal complexity, or with minimal block growth. They were introduced by Hedlund and Morse [MH40], and have been studied extensively.

For a sequence $(c_0, c_1, \dots)$ (finite or infinite) of 0's and 1's, let $C_{c_0,c_1,\dots} = \{x : T^i x \in A_{c_i} \text{ for all } i\}$. If $x \in C_{c_0,c_1\dots}$, then $(c_0, c_1, \dots)$ is a *coding* for the orbit of x (or a portion thereof, if the sequence is finite). Let Σ be the set of finite codings $c_0, c_1, \dots, c_n$ which actually occur, i.e. for which $C_{c_0,\dots,c_n} \neq \emptyset$. Let

$$V_j = \{x : x \in C_{c_0,\dots c_j} \text{ and such that } c_0, \dots c_j, 0 \text{ and } c_0, \dots c_j, 1 \in \Sigma\}.$$

This is the set of *'undetermined' points at step* j, that is, points whose coding up to step j does not determine the coding at step $j+1$.

We want to find asymptotics on how often a point is undetermined; specifically, we will prove

Theorem A. *For almost all* $x \in [0,1)$ *and almost all* α,

$$\lim_{n\to\infty} \frac{\log \sum_{j=1}^n \chi_{V_j}(x)}{\log \sum_{j=1}^n \lambda(V_j)} = 1.$$

As in [CC17], the full measure condition on α is a diophantine condition involving the continued fraction expansion of α. It will be stated explicitly in the proof.

To understand why Theorem A constitutes a shrinking target problem, consider the following. Let $\mathcal{P}_j = \vee_{k=0}^j R^k_\alpha \mathcal{P}$, the partition generated by $\mathcal{P}$ and its first j translates. For $x \in X$, denote by $[[x]]_j$ the atom of x in $\mathcal{P}_j$. The coding $c_0, \dots c_j$ determines only the atom $[[R^j_\alpha x]]_j$. A point x will belong to V_j if and only if $R^j_\alpha x$ is in $[[1-\alpha]]_j$ as the image of this atom under one more rotation contains points in both A_0 and A_1. We will denote $[[1-\alpha]]_j$ by U_j – these are the shrinking targets which we are trying to hit. Note that $U_j = R^j_\alpha(V_j)$.

The logarithms in Theorem A indicate a weaker asymptotic result than in [CC17]. The stronger version is not true:

Theorem B. *For almost all* α,

$$\lim_{N\to\infty} \frac{\sum_{j=1}^N \chi_{V_j}(x)}{\sum_{j=1}^N \lambda(V_j)}$$

does not exist for almost every $x \in [0,1)$.

Thus, Theorem A is in some sense the best one can hope for in this setting, an interesting contrast with the stronger results obtained for targets of the form $B(y, r_i)$.

1.2. **Notation and an outline of the paper.** The key tool throughout the paper is the continued fraction expansion of α and its close relationship to the dynamics of the rotation by α. Throughout, $\alpha \in (0,1)$ is assumed to be irrational. We write

$$\alpha = [0; a_1, a_2, a_3, \dots]$$

for the continued fraction expansion of α. Note that the *elements* a_i of the continued fraction depend on α; we will at times write $a_i(\alpha)$ to emphasize this dependence. The *convergents* to α are the rationals $\frac{p_k}{q_k}$. The k^{th} convergent is the best rational approximation to α with denominator $\leq q_k$. The q_k can be computed by the recurrence relation $q_{k+1} = a_{k+1}q_k + q_{k-1}; q_0 = 1, q_1 = a_1$.

We will prove Theorem B first, in Section 2. The almost sure existence of elements of the continued fraction expansion which are very large in relation to the preceding elements drives the argument.

Theorem A is proved in Section 3. There we prove a set of looser bounds on $\sum_{i=1}^{n} a_i$ and $\sum_{j=1}^{q_n} \chi_{V_j}(x)$ which hold for almost all α and which are sufficient for the statement of Theorem A.

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2. Failure of a stronger convergence

Before turning to the proof of Theorem A, we give an argument as to why there is no stronger theorem along the lines of convergence of

$$\frac{\sum_{j=1}^{n} \chi_{V_j}(x)}{\sum_{j=1}^{n} \lambda(V_j)}. \tag{1}$$

We begin with a proposition proving the existence of very large elements a_n for the continued fraction expansion and use this to show that, for very long stretches of time certain points are undetermined more often than $\sum_{j=1}^{n} \lambda(V_j)$ predicts. Namely, we will prove:

Proposition 2.1. *For any $C \in \mathbb{R}$ and almost every α there exist infinitely many m such that*

$$a_m > C \sum_{i=1}^{m-1} a_i. \tag{2}$$

We need a series of preliminary results to prove this. The following lemma appears in [Khi97, page 60].

Lemma 2.2. *For any $n, b_1, ..., b_n \in \mathbb{N}$ we have*

$$\frac{1}{3b_n^2} < \frac{\lambda(\{\alpha : a_1(\alpha) = b_1, ..., a_n(\alpha) = b_n\})}{\lambda(\{\alpha : a_1(\alpha) = b_1, ..., a_{n-1}(\alpha) = b_{n-1}\})} < \frac{2}{b_n^2}.$$

From this it is an easy exercise to deduce:

Corollary 2.3.

$$\frac{1}{3b_n} < \frac{\lambda(\{\alpha : a_1(\alpha) = b_1, ..., a_n(\alpha) \geq b_n\})}{\lambda(\{\alpha : a_1(\alpha) = b_1, ..., a_{n-1}(\alpha) = b_{n-1}\})} < \frac{4}{b_n}.$$

Let $W_n = \left\{\alpha : \sum_{i=1}^{n} a_i(\alpha) < 10n \log n\right\}$.

Lemma 2.4. $\lambda(W_n) > \frac{1}{10}$ *for* $n > 7$.

Proof. Let $A_n = \{\alpha : a_i(\alpha) < n^2 \text{ for all } i \leq n\}$. By Corollary 2.3, $\lambda(\{a_i(\alpha) \geq n^2\}) < \frac{4}{n^2}$ for any i. Thus, $\lambda(A_n^c) < \frac{4}{n}$.

Consider $\sum\limits_{i=1}^{n} \int_{A_n} a_i(\alpha) d\lambda$. We have,

$$\sum_{i=1}^{n} \int_{A_n} a_i(\alpha) d\lambda = \sum_{i=1}^{n} \sum_{j=1}^{n^2} j \cdot \lambda(A_n \cap \{a_i(\alpha) = j\})$$
$$< \sum_{i=1}^{n} \sum_{j=1}^{n^2} j \frac{2}{j^2}$$

where we have bounded $\lambda(A_n \cap \{a_i(\alpha) = j\}) < \frac{2}{j^2}$ using Lemma 2.2. The double sum is less than or equal to $2n(1 + \log n^2)$ which is bounded above by $5n \log n$, for $n > 7$.

Using Markov's inequality, we have

$$\lambda(W_n^c \cap A_n) \leq \frac{1}{10n \log n} \int_{\alpha \in A_n} \sum_{i=1}^{n} a_i(\alpha)\ d\lambda$$
$$\leq \frac{1}{10n \log n} 5n \log n$$
$$\leq \frac{1}{2}.$$

Since $\lambda(A_n) \geq 1 - \frac{4}{n}$, we conclude that $\lambda(W_n \cap A_n)$ is at least $\frac{1}{10}$ for $n > 7$.

□

Remark. The bound in Lemma 2.4 is not optimal, as is easily seen from the proof. We are only concerned to find some bound away from zero.

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. Fix $C > 0$. Corollary 2.3 and the definition of W_{m-1} imply that

$$\lambda\Big(\{\alpha : a_1(\alpha) = b_1, \ldots, a_{m-1}(\alpha) = b_{m-1} \text{ and } a_m(\alpha) \geq 10C(m-1)\log(m-1)\} \cap W_{m-1}\Big)$$
$$\geq \frac{\lambda(\{\alpha : a_1(\alpha) = b_1, \ldots, a_{m-1}(\alpha) = b_{m-1}\} \cap W_{m-1})}{30C(m-1)\log(m-1)}.$$

From this we have that

$$\lambda\Big(W_{m-1} \cap \{\alpha : a_m(\alpha) \geq 10C(m-1)\log(m-1)\}\Big) \geq \frac{\lambda(W_{m-1})}{30C(m-1)\log(m-1)}.$$

Let $G_m = W_{m-1} \cap \{\alpha : a_m \geq 10C(m-1)\log(m-1))\}$. Notice that $\alpha \in G_m$ implies that $a_m(\alpha) > C \sum_{i=1}^{m-1} a_i(\alpha)$. Then

$$\lambda(G_m) \geq \frac{\lambda(W_{m-1})}{30C(m-1)\log(m-1)} > \frac{1}{300C(m-1)\log(m-1)}.$$

Using this estimate,

$$\sum_{m=1}^{\infty} \lambda(G_m) > \sum_{m=1}^{\infty} \frac{1}{300C(m-1)\log(m-1)} = \infty.$$

To complete the proof we need two lemmas:

Lemma 2.5. *Let A_i be measurable subsets of a space with probability measure λ. If there exists $C > 0$ such that $\lambda(A_i \cap A_j) < C\lambda(A_i)\lambda(A_j)$ and $\sum_{i=1}^{\infty} \lambda(A_i) = \infty$, then $\lambda\left(\bigcap_{N=1}^{\infty} \bigcup_{i=N}^{\infty} A_i\right) > \frac{1}{4C} > 0$.*

Proof. Let $B_{N,M} = \cup_{i=N}^{M} A_i$. If $\sum_{i=N}^{M} \lambda(A_i) < \frac{1}{2C}$ then for any $j \notin [N, M]$ we have that

$$\lambda(A_j \setminus B_{N,M}) \geq \lambda(A_j) - \sum_{i=N}^{M} C\lambda(A_j)\lambda(A_i) > \frac{1}{2}\lambda(A_j).$$

Because $\sum \lambda(A_i) = \infty$, the above implies that $\lambda(B_{N,\infty}) \geq \frac{1}{4C}$ for all N. Because we are in a finite measure space it follows that $\lambda\left(\bigcap_{N=1}^{\infty} \bigcup_{i=N}^{\infty} A_i\right) = \lim_{N\to\infty} \lambda\left(\cup_{i=N}^{\infty} A_i\right)$ and so is at least $\frac{1}{4C}$. □

Lemma 2.6. *If $m > n > 7$ then $\lambda(G_m \cap G_n) \leq 120\lambda(G_m)\lambda(G_n)$.*

Proof. By Corollary 2.3, if $A_1 = \{\alpha : a_i(\alpha) = b_i \text{ for } 1 \leq i < m\}, A_2 = \{\alpha : a_i(\alpha) = c_i \text{ for } 1 \leq i < m\}$ are both subsets of W_{m-1} then

$$\frac{1}{12}\frac{\lambda(A_2 \cap G_m)}{\lambda(A_2)} \leq \frac{\lambda(A_1 \cap G_m)}{\lambda(A_1)} \leq 12\frac{\lambda(A_2 \cap G_m)}{\lambda(A_2)}. \tag{3}$$

Write $G_n = \sqcup_i A_i$ with each A_i of the form $A_i = \{\alpha : a_\ell(\alpha) = b_\ell, 1 \leq \ell < m\}$. Then $G_m \cap G_n = \sqcup_i (G_m \cap A_i)$, where we can assume all $A_i \subset W_{m-1}$. Then, using equation (3),

$$\begin{aligned}\lambda(G_m \cap G_m) &= \sum_i \lambda(G_m \cap A_i) \\ &\leq \sum_i 12\frac{\lambda(A^* \cap G_m)}{\lambda(A^*)}\lambda(A_i) \\ &\leq 12\lambda(G_n)\frac{\lambda(A^* \cap G_m)}{\lambda(A^*)}\end{aligned}$$

for an arbitrary $A^* \subset W_{m-1}$ of the form above. Since a subcollection of the A_i form a partition of W_{m-1}, by restricting the above estimate to that subcollection we have $\lambda(G_m \cap G_n) \leq 12\frac{1}{\lambda(W_{m-1})}\lambda(G_m)\lambda(G_n)$. The result follows, using Lemma 2.4. □

Applying these two lemmas we conclude that there is a positive measure set of α for which $a_m(\alpha) \geq C\sum_{i=1}^{m-1} a_i(\alpha)$ infinitely often. If α is in this set, its image under the Gauss map is as well, so by the ergodicity of that map the set of such α in fact has full measure. □

The following two lemmas on the shrinking targets U_j are also needed to complete our proof of non-convergence for sums like (1). Recall that $U_j = R_\alpha^j(V_j)$ and

$$V_j = \{x : x \in C_{c_0,\dots c_j} \text{ and such that } c_0, \dots c_j, 0 \text{ and } c_0, \dots c_j, 1 \in \Sigma\}.$$

These lemmas are proved using the partial fraction expansion of α. We will denote by $[y]$ the value modulo 1 of a real number y and by $\langle\langle y \rangle\rangle$ the distance from y to the nearest integer.

Lemma 2.7. *Let*

$$r_j = max\{q_k : q_k \leq j\}$$

$$s_j = max\{q_k : q_{k+1} \leq j\}$$

$$t_j = max\{T \in \mathbb{N} : s_j + T r_j \leq j\}.$$

Then

$$R_\alpha(U_j) = \big[[s_j\alpha] + t_j[r_j\alpha], [r_j\alpha]\big)$$

or

$$R_\alpha(U_j) = \big[[r_j\alpha], [s_j\alpha] - t_j(1 - [r_j\alpha])\big),$$

and

$$\lambda(U_j) = \lambda(V_j) = \langle\langle r_j\alpha \rangle\rangle + \langle\langle s_j\alpha \rangle\rangle - t_j\langle\langle r_j\alpha \rangle\rangle.$$

Remark. Note that if $r_j = q_k$, $s_j = q_{k-1}$ and $t_j < a_{k+1}$.

Proof. Note that $\langle\langle r_j\alpha \rangle\rangle$ is smaller than $\langle\langle i\alpha \rangle\rangle$ for all $i \leq j$.

Case 1: $0 < [r_j\alpha] < 1/2$. As the convergents alternate in approximating α from above and below, $1/2 < [s_j\alpha] < 1$. The only possible improvement in $[r_j\alpha]$ as an upper bound for $R_\alpha(U_j)$ would come from finding some l with $\langle\langle l\alpha \rangle\rangle < \langle\langle r_j\alpha \rangle\rangle$. This is not possible for $l \leq j$. Thus the upper endpoint of $R_\alpha(U_j)$ is $[r_j\alpha]$ as desired.

The lower bound on $R_\alpha(U_j)$ given by $[s_j\alpha]$ can be improved only by adding $[r_j\alpha]$ some number of times, as r_j is the only integer $\leq j$ with $\langle\langle r_j\alpha \rangle\rangle < \langle\langle s_j\alpha \rangle\rangle$. The lower endpoint will thus be of the form $y = [s_j\alpha] + T[r_j\alpha]$ and will be found by taking T as large as possible such that the $s_j + Tr_j$ rotations required to produce this point do not exceed j; this number is t_j.

We calculate that $\lambda(U_j) = \langle\langle r_j\alpha \rangle\rangle + (1 - [s_j\alpha] - t_j[r_j\alpha])$ using the fact that in this case $\langle\langle r_j\alpha \rangle\rangle = [r_j\alpha]$. Since $\langle\langle s_j\alpha \rangle\rangle = 1 - [s_j\alpha]$, this simplifies to the desired result.

Case 2: $1/2 < [r_j\alpha] < 1$. Then $0 < [s_j\alpha] < 1/2$ and the lower endpoint of $R_\alpha(U_j)$ is $[r_j\alpha]$. As before, the upper endpoint is of the form $[s_j\alpha] - T(1 - [r_j\alpha])$. The best such endpoint is found by taking T as large as possible, i.e. equal to t_j.

Finally, we calculate again

$$\begin{aligned}\lambda(U_j) =& \langle\langle s_j\alpha \rangle\rangle - t_j(1 - [r_j\alpha]) + (1 - [r_j\alpha])\\ =& \langle\langle s_j\alpha \rangle\rangle - t_j\langle\langle r_j\alpha \rangle\rangle + \langle\langle r_j\alpha \rangle\rangle.\end{aligned}$$

□

For use in the lemma below as well as later in the paper, we fix some notation. We will adopt interval notation ($[n, m)$, etc.) to denote intervals of integers; context will make the distinction between these and subsets of the real interval $[0, 1)$ clear.

We let $I_i = [q_i, q_{i+1})$. We let

$$J_b^i = \begin{cases} [q_i, q_{i-1} + q_i) & \text{if } b = 1, \\ [q_{i-1} + (b-1)q_i, q_{i-1} + bq_i) & \text{if } 1 < b \leq a_{i+1}. \end{cases}$$

Let $\mathcal{J}$ denote the collection of all the J_b^i's. We note that $J_b^i \subset I_i$ and that these intervals are disjoint.

Further, let $J_2^i = [q_{i-1} + q_i, q_{i-1} + 2q_i)$ for all i, whether $a_{i+1} \geq 2$ or not. If $a_{i+1} = 1$, $J_2^i \subset I_{i+1}$ and it equals J_1^{i+1}, but we note that in any case $\{J_2^i\}_{i \in \mathbb{N}}$ consists of pairwise disjoint intervals.

Lemma 2.8. *For any $J \in \mathcal{J}$, and for all $l \in J$, the sets $V_l = R_\alpha^{-l} U_l$ are pairwise disjoint.*

Proof. Fix $J_b^i \in \mathcal{J}$. For $l \in J_b^i$, Lemma 2.7 tells us that $R_\alpha U_l$ is the interval containing 0 bounded by $R_\alpha^{q_i}(0)$ and $R_\alpha^{q_{i-1}+(b-1)q_i}(0)$.

Suppose that $l > k \in J_b^i$. Then $U_l = U_k =: U$, and $R_\alpha^{-l} U \cap R_\alpha^{-k} U \neq \emptyset$ if and only if $R_\alpha U \cap R_\alpha^{l-k}(R_\alpha U) \neq \emptyset$. For such an intersection to occur, R_α^{l-k} of some endpoint of $R_\alpha U$ must lie in $R_\alpha U$.

We examine the two cases: $b = 1$ and $b > 1$.

If $b = 1$, $J_b^i = [q_i, q_{i-1} + q_i)$, $1 < l - k < q_{i-1}$, and the endpoints of $R_\alpha U$ are $R_\alpha^{q_i}(0)$ and $R_\alpha^{q_{i-1}}(0)$. The first time after q_{i-1} that the orbit of 0 hits U is $q_{i-1} + q_i$. But $(l - k) + q_{i-1} < q_{i-1} + q_i$ and $(l - k) + q_i < q_{i-1} + q_i$, so neither endpoint of $R_\alpha U$ will return to $R_\alpha U$ under R_α^{l-k}, proving the desired disjointness.

If $b > 1$, $J_b^i = [q_i + (b-1)q_i, q_{i-1} + bq_i)$, $1 < l - k < q_i$, and the endpoints of $R_\alpha U$ are $R_\alpha^{q_i}(0)$ and $R_\alpha^{q_{i-1}+(b-1)q_i}(0)$. The first time after $q_{i-1} + (b-1)q_i$ that the orbit of 0 hits U is $q_{i-1} + bq_i$. But $(l-k) + q_i < q_{i-1} + bq_i$ since $l - k < q_i$ and $b \geq 2$ and $(l - k) + (q_{i-1} + (b-1)q_i) < q_{i-1} + bq_i$ since $l - k < q_i$, so neither endpoint of $R_\alpha U$ will return to $R_\alpha U$ under R_α^{l-k}, again proving disjointness.

□

Corollary 2.9. For all m,

$$\sum_{j=1}^{q_m - 1} \lambda(V_j) < \sum_{i=1}^{m} a_i.$$

Proof. In $\mathcal{J}$, there are a_i intervals J_b^{i-1} contained in $I_{i-1} = [q_{i-1}, q_i)$. By Lemma 2.8, $\sum_{l \in J_b^{i-1}} \lambda(V_l) < 1$ since the V_l are disjoint over these indices. Thus, $\sum_{j=q_{i-1}}^{q_i - 1} \lambda(V_j) < a_i$ and the result follows. □

The following technical tool, a consequence of equidistribution of points under the rotation R_α and regularity of measures will be used in the proof of Theorem B:

Lemma 2.10. *Let $A \subset [0, 1)$ have positive measure and fix $\delta > 0$. Suppose we have families $\{X_m\}_{m \in \mathbb{N}}$ and $\{Y_m\}_{m \in \mathbb{N}}$ of subsets of $[0, 1)$ such that*

- *$\lambda(X_m), \lambda(Y_m) > \delta > 0$ for all m,*

- *For each m, $X_m = \bigcup_{k=1}^{K_m} R_\alpha^k(U_m)$ and $Y_m = \bigcup_{k=1}^{K_m} R_\alpha^k(V_m)$ where U_m and V_m are intervals and $K_m \to \infty$ as $m \to \infty$.*

Them, for any sufficiently large m, there exists a pair of points $x^* \in X_m \cap A$ and $y^* \in Y_m \cap A$ with $|x^* - y^*| < \delta$.

Proof. Choose a positive ϵ satisfying $\epsilon < \frac{(.99)\lambda(A)\delta}{2+(.99)\delta}$. This choice guarantees $(\frac{1}{2})(.99)(\lambda(A) - \epsilon)\delta > \epsilon$. Since A has finite measure, there is a finite, disjoint union of open intervals $B = \bigsqcup_{i=1}^{n} I_i$ such that $\lambda(A\Delta B) < \epsilon$. By the equidistribution of points under R_α and the fact that $K_m \to \infty$, we may pick $M > 0$ so large that for all $m > M$,

$$\lambda(I_i \cap X_m) > .99\lambda(I_i)\delta$$

$$\lambda(I_i \cap Y_m) > .99\lambda(I_i)\delta$$

for all $i = 1, \ldots, n$, using our lower bound on the measures of X_m and Y_m. Further pick M so large that for $m > M$, the maximum separation between two adjacent points in $\{R_\alpha^k 0\}_{k=1}^{K_m}$ is $< \delta$.

Consider the intervals forming X_m and Y_m which are contained in I_i. For each interval U which is a connected component of X_m, let V_U be its nearest neighbor to the right among the connected components of Y_m. (Such a neighbor exists for all but possibly the last such U contained in I_i. We may choose M so large that the number of X_m intervals in I_i is very large, making this exceptional subinterval's contribution to the argument below negligible.) Note that $\max_{x \in U, y \in V_U} |x - y| < \delta$. If a pair (x^*, y^*) as desired does not exist, then for each pair (U, V_U), at least one of U, V_U contains no points in A. Therefore, $\lambda(I_i \setminus A) > \frac{1}{2}(.99)\lambda(I_i)\delta$. Thus,

$$\begin{aligned} \lambda(B \setminus A) &> \sum_{i=1}^{n} \frac{1}{2}(.99)\lambda(I_i)\delta \\ &= \frac{1}{2}(.99)\lambda(B)\delta \\ &> \frac{1}{2}(.99)(\lambda(A) - \epsilon)\delta > \epsilon \end{aligned}$$

by our choice of ϵ. But this contradicts our choice of B, proving the lemma. □

To simplify notation a bit, we set for all integers m:

$$f_m(x) := \frac{\sum_{j=1}^{q_m-1} \chi_{V_j}(x)}{\sum_{j=1}^{q_m-1} \lambda(V_j)}.$$

Where it exists, we set

$$f(x) := \lim_{N\to\infty} \frac{\sum_{j=1}^{N} \chi_{V_j}(x)}{\sum_{j=1}^{N} \lambda(V_j)}.$$

Note that, where it exists, $\lim_{m\to\infty} f_m(x) = f(x)$ and f is measurable. In addition, by Fatou's Lemma, f will be integrable over the set where it is defined, since $\int_{[0,1)} f_m d\lambda = 1$ for all m. Therefore we can assume f takes only finite values.

We are now ready to prove Theorem B.

Proof of Theorem B. Fix $C > 0$ and apply Proposition 2.1 to find a full measure set of α satisfying equation (2) for infinitely many m. Fix any such m.

For all $b \in [2, a_m]$, let

$$W_b = \bigcup_{j \in J_b^{m-1}} V_j.$$

Note that by Lemma 2.8, this is a disjoint union, and using Lemma 2.7,

$$\lambda(W_b) = q_{m-1}\big[\langle\langle q_{m-2}\alpha\rangle\rangle - (b-2)\langle\langle q_{m-1}\alpha\rangle\rangle\big].$$

In addition, if $x \in W_b$, then it will belong to exactly one V_j with $j \in J_{b'}^{m-1}$ for all $b' \leq b$, and because $V_{j+q_k} \subset V_j$ for $q_k \leq j \leq q_{k+1} - q_k$,

$$\sum_{j=q_{m-1}}^{q_m-1} \chi_{V_j}(x) \geq b \quad \text{for all} \quad x \in W_b.$$

Choose any $\rho \in (1/8, 1/4)$ in such a way that $\rho a_m \in \mathbb{N}$ (possible since a_m is very large), and let $X_m = W_{\rho a_m}$. We then estimate the measure of X_m below using standard results on the convergents:

$$\begin{aligned}
\lambda(X_m) &= q_{m-1}\left[\langle\langle q_{m-2}\alpha\rangle\rangle - (\rho a_m - 1)\langle\langle q_{m-1}\alpha\rangle\rangle\right] \\
&\geq q_{m-1}\left[\frac{1}{q_{m-1}+q_{m-2}} - \rho a_m \frac{1}{q_m}\right] \\
&\geq \frac{1}{2} - \rho\frac{a_m q_{m-1}}{q_m} \\
&\geq \frac{1}{2} - \rho. \\
&\geq \frac{1}{4}.
\end{aligned}$$

Second, choose $\sigma \in (1/16, 1/8)$ so that $\sigma a_m \in \mathbb{N}$ and is ≥ 2. Let $Y_m = W_1 \backslash W_{\sigma a_m}$. Then, as any $y \in Y_m$ will not belong to V_j when $j \in J_b^m$ for $b \geq \sigma a_m$,

$$\sum_{j=q_{m-1}}^{q_m-1} \chi_{V_j}(y) \leq \sigma a_m - 1 \quad \text{for all} \quad y \in Y.$$

Using Corollary 2.9,

$$\sum_{j=1}^{q_m-1} \chi_{V_j}(y) \leq \sigma a_m - 1 + \sum_{i=1}^{m-1} a_i \text{ for all } y \in Y.$$

We can also estimate the measure of this set (recalling that a_m is very large):

$$\begin{aligned}
\lambda(Y_m) &= q_{m-1}\big[(\sigma a_m - 1)\langle\langle q_{m-1}\alpha\rangle\rangle\big] \\
&\geq q_{m-1}(\sigma a_m - 1)\frac{1}{q_m + q_{m-1}} \\
&\geq \frac{1}{2}\sigma\frac{a_m q_{m-1}}{2q_m} \\
&= \frac{1}{4}\sigma\frac{a_m q_{m-1}}{a_m q_{m-1} + q_{m-2}} \\
&\geq \frac{1}{4}\sigma\frac{a_m q_{m-1}}{(a_m+1)q_{m-1}} \\
&\geq \frac{1}{4}\sigma\frac{1}{2} \ \geq \frac{1}{128}.
\end{aligned}$$

Estimates here are certainly not precise; the key point is that X_m and Y_m have a positive lower bound on their measures which is independent of m. Let $\delta = \frac{1}{128}$.

Using the results above, for all $x \in X_m$ and $y \in Y_m$,

$$\begin{aligned}
\sum_{j=1}^{q_m-1}\chi_{V_j}(x) - \sum_{j=1}^{q_m-1}\chi_{V_j}(y) &> \rho a_m - \sigma a_m + 1 - \sum_{i=1}^{m-1} a_i \\
&> (\rho - \sigma - 1/C)a_m.
\end{aligned}$$

Finally, using Corollary 2.9,

$$\begin{aligned}
\big|f_m(x) - f_m(y)\big| = \frac{\Big|\sum_{j=1}^{q_m-1}\chi_{V_j}(x) - \sum_{j=1}^{q_m-1}\chi_{V_j}(y)\Big|}{\sum_{j=1}^{q_m-1}\lambda(V_j)} &\geq \frac{(\rho-\sigma-1/C)a_m}{\sum_{i=1}^m a_i} \\
&\geq \frac{(\rho-\sigma-1/C)a_m}{(1+1/C)a_m} \\
&= \frac{(\rho-\sigma-1/C)}{(1+1/C)}.
\end{aligned}$$

By choosing C sufficiently large, and since $\rho > \sigma$, we have $|f_m(x) - f_m(y)| \geq D > 0$ for all m such that equation (2) holds and all $x \in X_m$, $y \in Y_m$.

Let $Z = \{x : f(x) \text{ exists}\}$. Towards a contradiction, assume $\lambda(Z) > 0$. Fix $\epsilon < \frac{D}{3}$ and $< \frac{\lambda(Z)}{2}$.

Since f is measurable, by Luzin's Theorem there is a compact set $G \subset Z$ with $\lambda(G) > \lambda(Z) - \epsilon$ over which f is (uniformly) continuous. Let $\delta > 0$ be such that $|x - y| < \delta$ and $x, y \in G$ imply $|f(x) - f(y)| < \epsilon$.

Let

$$Z_N = \Big\{x \in Z : \text{ for all } n \geq N, \frac{\sum_{j=1}^n \chi_{V_j}(x)}{\sum_{j=1}^n \lambda(V_j)} \text{ is within } \epsilon \text{ of } f(x)\Big\}.$$

Under our assumption $\lambda(Z_N) \to \lambda(Z)$ as $N \to \infty$. Pick N_0 so large that $\lambda(Z_{N_0}) > \lambda(Z) - \epsilon$ and, therefore, $\lambda(G \cap Z_{N_0}) > \lambda(Z) - 2\epsilon > 0$ by the choice of ϵ.

Let m be chosen so large that the following hold:

- $q_m > N_0$,
- a_m satisfies condition (2), and
- $\{X_m\}_{m\in\mathbb{N}}$ and $\{Y_m\}_{m\in\mathbb{N}}$ satisfy Lemma 2.10 with $A = G \cap Z_{N_0}$.

Then we may take $x^* \in X_m \cap G \cap Z_{N_0}$ and $y^* \in Y_m \cap G \cap Z_{N_0}$ with $|x^* - y^*| < \delta$. As $x^*, y^* \in Z_{N_0}$ and $q_m > N_0$, $|f_m(x^*) - f(x^*)| < \epsilon$ and $|f_m(y^*) - f(y^*)| < \epsilon$. As both points are in G and $|x^* - y^*| < \delta$, $|f(x^*) - f(y^*)| < \epsilon$. We conclude that $|f_m(x^*) - f_m(y^*)| < 3\epsilon < D$. But this contradicts our result above on the minimum difference between the values of f_m at points in X_m and Y_m when a_m satisfies (2). Therefore there is a set of full measure where the f_m do not converge, completing the proof.

□

3. Proof of Theorem A

Towards Theorem A, we claim the following set of inequalities:

There exists a positive constant C_1 such that for almost every α and $x \in [0, 1)$,

$$C_1 n(\log n)^3 > \sum_{i=1}^{n} a_i(\alpha) \geq \sum_{j=1}^{q_n - 1} \chi_{V_j}(x) > \frac{1}{4}(n-2). \tag{4}$$

The middle inequality follows from almost the same proof as Corollary 2.9. We prove the other two inequalities in the following sequence of Lemmas. Lemma 3.1 specifies the full measure set of α for which we prove Theorem A.

Lemma 3.1. *There exists a positive constant C_1 such that for almost every α, $C_1 n(\log n)^3 > \sum_{i=1}^{n} a_i(\alpha)$ for all $n > 7$.*

Proof. As in the proof of Lemma 2.4, set $A_n = \{\alpha : a_i(\alpha) < n^2 \text{ for all } i \leq n\}$. As before, $\int_{A_n} \sum_{i=1}^{n} a_i(\alpha) d\lambda(\alpha) \leq 5n \log n$ (for $n > 7$). Note also that λ-a.e. α belongs to A_n for all but finitely many n. It follows from Markov's inequality that

$$\lambda\left(\{\alpha \in A_n : \sum_{i=1}^{n} a_i(\alpha) > 10n(\log n)^{2.1}\}\right) \leq \frac{1}{10n(\log n)^{2.1}} \int_{A_n} \sum_{i=1}^{n} a_i(\alpha) d\lambda$$
$$\leq \frac{1}{2}\left(\frac{1}{\log n}\right)^{1.1}.$$

Since almost every α belongs to A_n for all but finitely many n, almost every α belongs to A_{10^k} for all but finitely many k. Then

$$\lambda\left(\left\{\alpha \in A_{10^k} : \sum_{i=1}^{10^k} a_i(\alpha) > 10^{k+1}(\log 10^k)^{2.1}\right\}\right) \leq \left(\frac{1}{\log 10^k}\right)^{1.1}.$$

These measures form a summable sequence, so for a.e. α,

$$\sum_{i=1}^{10^k} a_i(\alpha) \leq 10^{k+1} \left(\log 10^k\right)^{2.1} \text{ for all but finitely many } k.$$

This implies the Lemma because for large enough k, we have $10^k(\log 10^k)^3 \geq 10^{k+1}(\log 10^{k+1})^{2.1}$.

□

We will give a lower bound on $\sum_{j=1}^{q_n} \chi_{V_j}(x)$ by bounding below the sum over the J_2^i. As we noted above, $\{J_2^i\}_{i\in\mathbb{N}}$ is a disjoint set of intervals. Let

$$h_i(x) = \sum_{j\in J_2^i} \chi_{V_j}(x).$$

Lemma 3.2. *For all i,*

$$\int_{[0,1)} h_i(x)d\lambda > 1/2.$$

Proof. As per Lemma 2.8, over $j \in J_2^i$, the V_j are disjoint, so $h_i(x) \in \{0,1\}$. The length of the interval J_2^i is q_i, and for $j \in J_2^i$,

$$\lambda(V_j) = \langle\langle q_{i-1}\alpha\rangle\rangle,$$

using the description of $R_\alpha(U_j)$ provided by Lemma 2.7. By Theorem 13 in [Khi97], $\langle\langle q_{i-1}\alpha\rangle\rangle > \frac{1}{q_{i-1}+q_i}$. We may then bound the integral from below by

$$\int_{[0,1)} h_i(x)d\lambda > \frac{q_i}{q_i+q_{i-1}} > \frac{q_i}{2q_i} = \frac{1}{2}.$$

□

The following sequence of results prove that the random variables $h_i(x)$ are (approximately) independent.

Lemma 3.3. *Let $[c,d) \subset [0,1)$. Let $f_{[c,d)}(i,b) = \#\{[c,d) \cap \cup_{l\in J_b^i} R_\alpha^{-l}(0)\}$. Then*

$$\lambda([c,d))\left|J_b^i\right| - 2 \le f_{[c,d)}(i,b) \le \lambda([c,d))\left|J_b^i\right| + 2.$$

Proof. By Theorem 1 in [Kes66], each interval $(\frac{j}{q_m}, \frac{j+1}{q_m})$ for $j = 0,1,\dots q_m - 1$ contains exactly one point of $R_\alpha^{-l}(0)$ with $1 \le l \le q_i$. Likewise, if $a = \min J_b^i$, each $I_j := R_\alpha^{-(a-1)}(\frac{j}{q_m}, \frac{j+1}{q_m})$ contains exactly one point of $R^{-l}(0)$ for $l \in J_b^i$.

$|J_b^i| = q_m$ where $m = i-1$ if $b=1$ and $m = i$ if $b > 1$. At least $\lambda([c,d))|J_b^i| - 2$ of the I_j above are completely contained in $[c,d)$, and at most $\lambda([c,d))|J_b^i| + 2$ of them intersect $[c,d)$. The result then follows. □

Proposition 3.4. *Fix k. For all i such that $q_i > k$ and any $1 \le b \le a_{i+1}$,*

$$\left(\frac{\lambda(V_k)|J_b^i| - 3}{\lambda(V_k)|J_b^i|}\right)\lambda(V_k)\lambda\left(\cup_{l\in J_b^i} V_l\right) \le \lambda\left(V_k \cap \bigcup_{l\in J_b^i} V_l\right) \le \left(\frac{\lambda(V_k)|J_b^i| + 3}{\lambda(V_k)|J_b^i|}\right)\lambda(V_k)\lambda\left(\cup_{l\in J_b^i} V_l\right).$$

Proof. Fix k. Let i be so large that $q_i > k$. By the previous lemma, the interval V_k is hit by the left endpoints of the V_l between $\lambda(V_k)|J_b^i| - 2$ and $\lambda(V_k)|J_b^i| + 2$ times.

As the sets V_l are disjoint and of the same measure over $l \in J_b^i$, this easily yields

$$\left(\lambda(V_k)|J_b^i| - 3\right)\lambda(V_{l^*}) \leq \lambda\left(V_k \cap \bigcup_{l\in J_b^i} V_l\right) \leq \left(\lambda(V_k)|J_b^i| + 3\right)\lambda(V_{l^*}) \text{ for any } l^* \in J_b^i.$$

Furthermore, for any $l^* \in J_b^i$, $|J_b^i|\lambda(V_{l^*}) = \lambda(\cup_{l\in J_b^i} V_l)$. Translating to an inequality with multiplicative errors yields

$$\begin{aligned}\left(\frac{\lambda(V_k)|J_b^i| - 3}{\lambda(V_k)|J_b^i|}\right)\lambda(V_k)\lambda\left(\cup_{l\in J_b^i} V_l\right) \\ \leq \lambda\left(V_k \cap \bigcup_{l\in J_b^i} V_l\right) \\ \leq \left(\frac{\lambda(V_k)|J_b^i| + 3}{\lambda(V_k)|J_b^i|}\right)\lambda(V_k)\lambda\left(\cup_{l\in J_b^i} V_l\right).\end{aligned}$$

□

Proposition 3.4 asserts near independence of the events V_k and $\cup_{l\in J_b^i} V_l$. Using it for all $k \in J_{b'}^j$ where $j < i$ (which guarantees $k < q_i$) we get the following corollary. It relates to calculating the correlation between a point being undetermined in the intervals $J_{b'}^j$ and J_b^i.

Corollary 3.5. For any $k \in J_{b'}^i$, and $J_{b'}^i, J_b^j$ disjoint, $j > i$,

$$\begin{aligned}\left(\frac{\lambda(V_k)|J_b^j| - 3}{\lambda(V_k)|J_b^j|}\right)\lambda\left(\cup_{k\in J_{b'}^i} V_k\right)\lambda\left(\cup_{l\in J_b^j} V_l\right) \\ \leq \lambda\left(\bigcup_{k\in J_{b'}^i} V_k \cap \bigcup_{l\in J_b^j} V_l\right) \\ \leq \left(\frac{\lambda(V_k)|J_b^j| + 3}{\lambda(V_k)|J_b^j|}\right)\lambda\left(\cup_{k\in J_{b'}^i} V_k\right)\lambda\left(\cup_{l\in J_b^j} V_l\right).\end{aligned}$$

Proof. This follows from summing Proposition 3.4's inequalities over the disjoint sets V_k for $k \in J_{b'}^i$. (The desire to compute this sum explains our preference for the formulation in terms of multiplicative bounds above.) □

Proposition 3.6. *For $j > i$*

$$\left(1 - \frac{3q_{i+1}}{q_j}\right)\int h_i d\lambda \int h_j d\lambda \leq \int h_i h_j d\lambda \leq \left(1 + \frac{3q_{i+1}}{q_j}\right)\int h_i d\lambda \int h_j d\lambda.$$

Proof. First,

$$\int h_i(x)h_j(x)d\lambda = \int\left(\sum_{l\in J_2^i}\chi_{V_l}(x)\right)\left(\sum_{l\in J_2^j}\chi_{V_l}(x)\right)d\lambda.$$

As over J_2^i and over J_2^j the sets V_l are disjoint, the integrand of the above has value 0 or 1 according to whether $x \in \left(\cup_{l\in J_2^i} V_l\right) \cap \left(\cup_{l\in J_2^j} V_l\right)$. Thus,

$$\int h_i h_j d\lambda = \lambda\left(\bigcup_{l\in J_2^i} V_l \cap \bigcup_{l\in J_2^j} V_l\right).$$

By Corollary 3.5, for $l \in J_2^i$ we get

$$\left(\frac{\lambda(V_l)|J_2^j|-3}{\lambda(V_l)|J_2^j|}\right)\lambda\left(\cup_{l\in J_2^i} V_l\right)\lambda\left(\cup_{l\in J_2^j} V_l\right)$$
$$\leq \lambda\left(\bigcup_{l\in J_2^i} V_l \cap \bigcup_{l\in J_2^j} V_l\right)$$
$$\leq \left(\frac{\lambda(V_l)|J_2^j|+3}{\lambda(V_l)|J_2^j|}\right)\lambda\left(\cup_{l\in J_2^i} V_l\right)\lambda\left(\cup_{l\in J_2^j} V_l\right)$$

To assess the value of the terms $\left(1 \pm \frac{3}{\lambda(V_l)|J_2^j|}\right)$ consider an arbitrary $l \in J_2^i$. As $U_l = R_\alpha V_l$, using the description of $R_\alpha U_l$ given by Proposition 2.7 and [Khi97, Theorem 13], $\lambda(V_l) > \langle\langle q_i\alpha\rangle\rangle > \frac{1}{q_{i-1}+q_i} \geq \frac{1}{q_{i+1}}$. From its description, $|J_2^j| = q_j$. Using these two bounds, $\frac{3}{\lambda(V_l)|J_2^j|} < \frac{3q_{i+1}}{q_j}$.

Returning to our inequalities for $\int h_i h_j$, as the V_l are disjoint over J_2^j or J_2^i we can translate back into integrals as so:

$$\left(1-\frac{3q_{i+1}}{q_j}\right)\int \sum_{l\in J_2^i}\chi_{V_l}(x)d\lambda \int \sum_{l\in J_2^j}\chi_{V_l}(x)d\lambda$$
$$\leq \int h_i h_j d\lambda \leq$$
$$\left(1+\frac{3q_{i+1}}{q_j}\right)\int \sum_{l\in J_2^i}\chi_{V_l}(x)d\lambda \int \sum_{l\in J_2^j}\chi_{V_l}(x)d\lambda.$$

These are the desired bounds on $\int h_i h_j d\lambda$. □

The independence result we want is the following.

Proposition 3.7. *There exist constants $C, b > 0$ such that*

$$\left|\int_{[0,1)} h_i(x)h_j(x)d\lambda - \int_{[0,1)} h_i(x)d\lambda \int_{[0,1)} h_j(x)d\lambda\right| < Ce^{-b|i-j|}.$$

Proof. We may assume $j > i$. Using Proposition 3.6, we need to show that the expression

$$\frac{3q_{i+1}}{q_j}\int h_i d\lambda \int h_j d\lambda$$

decays exponentially in $|i-j|$. A clear upper bound on each of $\int h_i d\lambda, \int h_j d\lambda$ is 1. As $q_{k+2} > 2q_k$, $\frac{q_{i+1}}{q_j}$ decays exponentially in $|i-j|$, as desired. □

We can apply this approximate independence to prove the remaining inequality in equation (4). Let $\tilde{h}_i(x) = h_i(x) - \int h_i(x)d\lambda$, and note that $\tilde{h}_i(x) \in (-1,1)$. Let $\tilde{s}_n(x) = \sum_{i=1}^{n} \tilde{h}_i(x)$.

Proposition 3.8. *For almost every $x \in S^1$, for sufficiently large n,*

$$\sum_{j=1}^{q_n - 1} \chi_{V_j}(x) > \frac{1}{4}(n-2).$$

Proof. First, for all $x \in [0,1)$, $\sum_{j=1}^{q_n-1} \chi_{V_j}(x) \geq \sum_{i=1}^{n-2} h_i(x)$ as $j \in J_2^i$ implies $j < q_{i+2}$.

Consider $\sum_{i=1}^{n-2} \int h_i(x)d\lambda$. By Lemma 3.2 this is bounded below by $\frac{1}{2}(n-2)$; it is bounded above by n as h_i takes only 1 or 0 as a value. Applying Chebyshev's inequality to $\tilde{s}_n$ yields (for any $\epsilon > 0$)

$$\begin{aligned} \lambda\left(\{x : |\tilde{s}_{n-2}(x)| > \epsilon(n-2)\}\right) &< \frac{\int \tilde{s}_{n-2}^2(x)d\lambda}{\epsilon^2(n-2)^2} \\ &= \frac{\sum_{i=1}^{n-2} \int \tilde{h}_i^2(x)d\lambda + 2\sum_{i<j} \int \tilde{h}_i(x)\tilde{h}_j(x)d\lambda}{\epsilon^2(n-2)^2} \\ &< \frac{D}{\epsilon^2(n-2)}. \end{aligned}$$

For the last inequality we have used the facts that $\tilde{h}_i(x) \in (-1,1)$ so $\sum_{i=1}^{n-2} \int \tilde{h}_i^2(x)d\lambda < n-2$ and that for some positive constant D, $2\sum_{i<j} \int \tilde{h}_i\tilde{h}_j d\lambda < (D-1)(n-2)$ by Proposition 3.7.

We restrict our attention to the subsequence of times $\{(n-2)^2\}$, obtaining

$$\lambda(\{x : |\tilde{s}_{(n-2)^2}(x)| > \epsilon(n-2)^2\}) < \frac{D}{\epsilon^2(n-2)^2}.$$

Summing the term on the right-hand side of the above inequality over all n yields a convergent series so by the Borel-Cantelli Lemma, for almost every $x \in [0,1)$,

$$\frac{\tilde{s}_{(n-2)^2}(x)}{(n-2)^2} \to 0 \qquad \text{as } n \to \infty.$$

Consider now the intervals $[(n-2)^2, (n-1)^2)$. As $\tilde{h}_i(x) \in (-1,1)$, for $k \in [(n-2)^2, (n-1)^2)$,

$$|\tilde{s}_{(n-2)^2}(x) - \tilde{s}_k(x)| < 2(n-2) + 1$$

so

$$\frac{|\tilde{s}_k(x)|}{k} < \frac{|\tilde{s}_{(n-2)^2}(x)| + 2(n-2) + 1}{k} \leq \frac{|\tilde{s}_{(n-2)^2}(x)| + 2(n-2) + 1}{(n-2)^2} \to 0$$

as $k \to \infty$.

We have now that for almost all x,

$$\frac{\sum_{i=1}^{n-2} h_i(x) - \int h_i(x)d\lambda}{n-2} \to 0.$$

As $\sum_{i=1}^{n-2}\int h_i(x)d\lambda \in (\frac{1}{2}(n-2),(n-2))$, for sufficiently large n, $\sum_{i=1}^{n-2} h_i(x) > \frac{1}{4}(n-2)$ as desired. □

We now prove a similar series of inequalities for $\sum_{j=1}^{q_n}\lambda(V_j)$, namely:

$$C_1 n(\log n)^3 > \sum_{i=1}^{n} a_i(\alpha) > \sum_{j=1}^{q_n-1}\lambda(V_j) > \frac{1}{2}(n-2). \tag{5}$$

The left-most inequality is Lemma 3.1 and the next is Corollary 2.9. It remains only to prove:

Lemma 3.9. *For all α,*

$$\sum_{j=1}^{q_n-1}\lambda(V_j) > \frac{1}{2}(n-2).$$

Proof. This follows easily from Lemma 3.2 after noting that

$$\sum_{j=1}^{q_n-1}\lambda(V_j) > \sum_{i=1}^{n-2}\sum_{j\in J_2^i}\lambda(V_j) = \sum_{i=1}^{n-2}\int_{[0,1)} h_i(x)d\lambda.$$

□

The inequalities collected above enable us to prove the main theorem:

Proof of Theorem A. Consider the full measure set of α satisfying Lemma 3.1. Suppose $n \in [q_m, q_{m+1})$. Then we have the following for almost every x:

$$\frac{1}{4}(m-2) < \sum_{j=1}^{q_m-1}\chi_{V_j}(x) \le \sum_{j=1}^{n}\chi_{V_j}(x) \le \sum_{j=1}^{q_{m+1}}\chi_{V_j}(x) < C_1(m+1)(\log(m+1))^3$$

$$\frac{1}{2}(m-2) < \sum_{j=1}^{q_m-1}\lambda(V_j) \le \sum_{j=1}^{n}\lambda(V_j) \le \sum_{j=1}^{q_{m+1}}\lambda(V_j) < C_1(m+1)(\log(m+1))^3$$

Taking logs and forming the relevant quotient, we see that the $\log(m-2)$ and $\log(m+1)$ terms dominate the $\log(\mathit{constant})$ and $\log(\log(-))$ terms. As $\frac{\log(m)}{\log(m+1)}$ and $\frac{\log(m+1)}{\log(m-2)} \to 1$, the result follows.

□

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