

ON THE HAUSDORFF DIMENSION OF $CAT(\kappa)$ SURFACES

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ABSTRACT. We prove that a closed surface with a $CAT(\kappa)$ metric has Hausdorff dimension = 2, and that there are uniform upper and lower bounds on the two-dimensional Hausdorff measure of small metric balls. We also discuss a connection between this uniformity condition and some results on the dynamics of the geodesic flow for such surfaces. Finally, we give a short proof of topological entropy rigidity for geodesic flow on certain $CAT(-1)$ manifolds.

1. INTRODUCTION

Let Σ be a closed surface, κ any real number, and let d be a locally $CAT(\kappa)$ metric on Σ . One quantity of natural interest is the Hausdorff dimension of (Σ, d) , denoted $\dim_H(\Sigma, d)$. This dimension is bounded below by 2, the topological (covering) dimension of Σ . However, for an arbitrary metric on Σ there is no upper bound; this can be seen by “snowflaking” the metric – replacing $d(x, y)$ with $d'(x, y) = d(x, y)^\alpha$ for $0 < \alpha < 1$, which raises the Hausdorff dimension by a factor of $1/\alpha$ (see, e.g. [TW05]). In this paper we examine the restriction placed on $\dim_H(\Sigma, d)$ by the $CAT(\kappa)$ condition, and prove the following theorem:

Theorem 1. *Let (Σ, d) be a $CAT(\kappa)$ closed surface. Then $\dim_H(\Sigma, d) = 2$. Moreover, there exists some $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$,*

$$\inf_{p \in \Sigma} H^2(B(p, \delta)) > 0 \quad \text{and} \quad \sup_{p \in \Sigma} H^2(B(p, \delta)) < \infty$$

where H^2 denotes the 2-dimensional Hausdorff measure and $B(p, \delta)$ is the ball of radius δ around p .

We note that the second statement of the theorem implies the first, but not vice versa. Indeed there are metric spaces with Hausdorff dimension d whose d -dimensional Hausdorff measures are zero or infinite. We became interested in this question for Hausdorff measures, in particular the uniform bounds on the measures of balls, while thinking about some results on entropy for geodesic flows on locally $CAT(-1)$ manifolds. As an immediate application of Theorem 1, we have

Corollary 2. *Let (Σ, d) be a closed surface with a $CAT(0)$ metric. Let ϕ_t be the geodesic flow on the space of geodesics for (Σ, d) . Then the topological entropy for the flow equals the volume growth entropy for the Hausdorff 2-measure.*

Corollary 2 is a version of Manning’s [Man79] analogous result for Riemannian manifolds of non-positive curvature, and relies on some work of Leuzinger [Leu06]. This is discussed in the final section of the paper, where we also establish the following entropy rigidity result for the geodesic flow.

Theorem 3. *Let (X, d) be a closed CAT(-1) manifold (not necessarily Riemannian), and suppose that X admits a Riemannian metric g so that (X, g) is a locally symmetric space. Let $h_{top}(\phi_t^d)$ and $h_{top}(\phi_t^g)$ be the topological entropies for the geodesic flows under the two metrics. Then*

$$h_{top}(\phi_t^d) \geq h_{top}(\phi_t^g)$$

and if equality holds, (X, d) is also locally symmetric. If $\dim X > 2$, (X, d) and (X, g) are isometric.

Theorem 3 is a reformulation of a rigidity result of Bourdon ([Bou96]). Our main observation is to note how, using Leuzinger's work [Leu06], Bourdon's theorem can be recast as a topological entropy rigidity theorem. This fact may well be known to experts, but we have not found it addressed in the literature.

The paper is organized as follows. In Sections 2 and 3 we show that small distance spheres around each point in Σ are topological circles, and that they are rectifiable with bounded length. In Section 4 we prove Theorem 1, and in Section 5 we discuss the extension of Theorem 1 to higher dimensions, and give an example which indicates some of the complications in doing so. In Section 6 we give the proof of topological entropy rigidity (Theorem 3) for certain locally CAT(-1) manifolds.

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2. THE TOPOLOGY OF SMALL DISTANCE SPHERES

Let $S(p, \epsilon) = \{z \in \Sigma : d(p, z) = \epsilon\}$ and $B(p, \epsilon) = \{z \in \Sigma : d(p, z) \leq \epsilon\}$ respectively denote the metric ϵ -sphere and ϵ -ball centered at p . In this section we prove for small ϵ , all $S(p, \epsilon)$ are topological circles. We note that the argument only works for surfaces. In Section 5 we give examples of higher-dimensional CAT(-1) manifolds where the analogous statement is not true – small metric spheres need not be topological spheres.

Throughout this section, we work at small scale. We fix $\epsilon_0 > 0$ small enough so that the following two conditions are satisfied:

- $\epsilon_0 \leq D_\kappa/2$ where D_κ is the diameter of the model space of constant curvature κ , and
- For all $p \in \Sigma$, $B(p, \epsilon_0)$ is (globally) CAT(κ).

At these scales, $B(p, \epsilon_0)$ is locally uniquely geodesic – in particular there is a unique geodesic from p to any point in $B(p, \epsilon_0)$, which varies continuously with respect to the endpoints. This will be a key fact in the work below. As a consequence, each such ball $B(p, \epsilon_0)$ is contractible, hence lifts isometrically to the universal cover $(\tilde{\Sigma}, \tilde{d})$.

The following Lemma will be useful. Its proof, which is straightforward and can be adapted to any dimension, can be found in [BH99, Proposition II.5.12].

Lemma 4. *Let $[xy]$ be a geodesic segment in Σ connecting an arbitrary pair of points x and y . Then $[xy]$ can be extended beyond y . That is, there is a geodesic segment (not necessarily unique) $[xy']$ properly containing $[xy]$ as its initial segment.*

Remark. Using the compactness of X , and a connectedness argument on \mathbb{R} , this lemma implies that each geodesic segment $[xy]$ can be infinitely extended.

The main result of this section is the following:

Proposition 5. *Let Σ be a complete $\text{CAT}(\kappa)$ surface. Then for all $\epsilon < \epsilon_0$, $S(p, \epsilon)$ is homeomorphic to the circle \mathbb{S}^1 .*

In order to establish this result, we use a well-known characterization of the circle \mathbb{S}^1 . The circle is the only compact, connected, metric space (X, d) with the property that for any pair of distinct points $a, b \in X$, the complement $X \setminus \{a, b\}$ is disconnected (see, e.g. [HY88, Theorem 2-28]). Let $p \in \Sigma$ be an arbitrary point in Σ , and to simplify notation, we set $S_\epsilon := S(p, \epsilon)$. We now claim that for $\epsilon < \epsilon_0$, S_ϵ is homeomorphic to a circle.

Lemma 6. *For all $\epsilon < \epsilon_0$, S_ϵ is a compact, path-connected, metric space.*

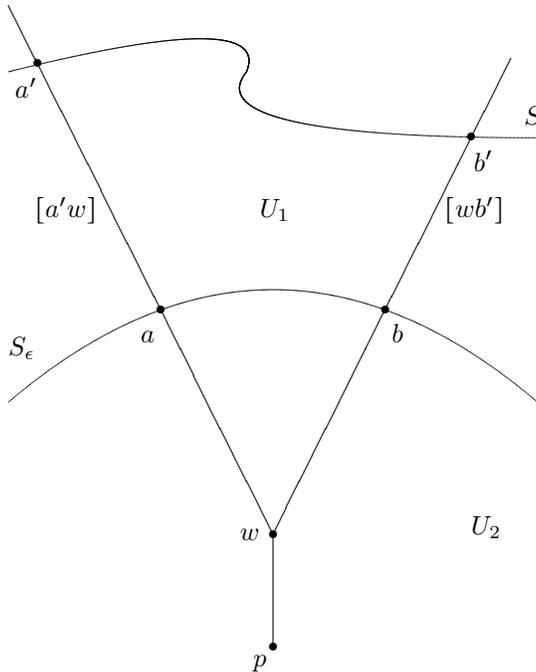
Proof. S_ϵ is a closed subset of the compact metric space Σ , so it is compact and metric. Since $\epsilon < \epsilon_0$, S_ϵ lifts homeomorphically to a subset of $\tilde{\Sigma}$. Since Σ is a surface, $\tilde{\Sigma}$ is homeomorphic to \mathbb{R}^2 or \mathbb{S}^2 , so we may take S_ϵ to be a compact subset of \mathbb{R}^2 or \mathbb{S}^2 .

S_ϵ has diameter $< D_\kappa$, so we may find a path S in \mathbb{R}^2 or \mathbb{S}^2 homeomorphic to \mathbb{S}^1 bounding a disk containing S_ϵ and remaining in $B(p, \epsilon_0)$. Let $\text{proj} : S \rightarrow S_\epsilon$ be the nearest point projection (for the $\text{CAT}(\kappa)$ metric lifted from Σ). This is well-defined, since each point in $B(p, \epsilon_0)$ has a unique geodesic connecting it to p , which intersects S_ϵ in a unique point. Moreover, since these geodesics vary continuously, proj is a continuous map. Since geodesics in $\tilde{\Sigma}$ are infinitely extendible, for any point z on S_ϵ , the geodesic segment $[pz]$ extends to a geodesic which hits S at a point q . Then $\text{proj}(q) = z$ and so this map is also surjective. The surjective, continuous map from the path-connected set S to S_ϵ proves that the latter is path-connected. \square

Lemma 7. *For all $\epsilon < \epsilon_0$, and pairs of distinct points $\{a, b\} \subset S_\epsilon$, the space $S_\epsilon \setminus \{a, b\}$ is disconnected.*

Proof. As in the proof of Lemma 6, let S in \mathbb{R}^2 or \mathbb{S}^2 be a Jordan curve containing S_ϵ and remaining in $B(p, \epsilon_0)$. In particular, from the Schoenflies theorem, we can now view S_ϵ as contained inside a closed topological disk \mathbb{D}^2 (the curve S along with its interior). Given the two distinct points $a, b \in S_\epsilon$, extend the two geodesic segments $[pa], [pb]$ to geodesic segments $[pa'], [pb']$, where a', b' are a pair of (distinct) points on S , and $[pa], [pb]$ are both contained in the interior of \mathbb{D}^2 .

Note that $[pa'], [pb']$ might coincide on some subgeodesic originating from p . Let w be the point at which these two geodesic segments separate, and consider the concatenation $[a'w]$ with $[wb']$. By uniqueness of geodesics in $B(p, \epsilon_0)$, we have that $[a'w] \cap [wb'] = \{w\}$, and hence they concatenate to give an embedded arc joining the pair of points $a', b' \in S = \partial\mathbb{D}^2$. Moreover, the interior $(a'b')$ of the arc is contained in the interior of \mathbb{D}^2 . It follows that $(a'b')$ separates the interior of the disk into two connected components U_1, U_2 . (See Figure 1.)

FIGURE 1. Proving S_ϵ is a circle.

Now by way of contradiction, let us assume $S_\epsilon \setminus \{a, b\}$ is connected. Then without loss of generality, $S_\epsilon \cap U_1$ must be empty. On the other hand, U_1 is homeomorphic to an open disk, whose boundary is a Jordan curve (formed by the arc $(a'b')$ in the interior of \mathbb{D}^2 , along with the portion of the boundary S joining a' to b'). The boundary of U_1 contains the arc (wa') passing through a , and the distance to p varies continuously along (wa') from a number $< \epsilon$ (since $w \neq a$) to a number $> \epsilon$ (since $a \neq a'$). Pick an arc η inside U_1 joining w to a' , and consider the distance function restricted to η . It varies continuously from $< \epsilon$ to $> \epsilon$, but since $S_\epsilon \cap U_1 = \emptyset$, is never equal to ϵ . This is a contradiction, completing the proof. \square

Using the topological characterization of \mathbb{S}^1 , Proposition 5 now follows immediately from Lemma 6 and Lemma 7.

3. THE GEOMETRY OF SMALL DISTANCE SPHERES

We now want to find a $0 < \delta_0 < \epsilon_0$ such that, for all $p \in \Sigma$ and $\epsilon < \delta_0$, S_ϵ is rectifiable with length uniformly bounded above. Let us denote by $l(\gamma)$ the length of a curve γ , where l takes the value ∞ if the curve is not rectifiable.

Lemma 8. *Let Σ be a $CAT(\kappa)$ surface. Then for each $p \in \Sigma$ there exists some $0 < \epsilon < \epsilon_0$ such that $S(p, \epsilon)$ is a rectifiable curve.*

Proof. Fix p . By Proposition 5, for $\epsilon < \epsilon_0$, S_ϵ is homeomorphic to a circle. By way of contradiction, let us assume that for all $n \in \mathbb{N}$, $S_{\epsilon/n}$ is not rectifiable, i.e. that $l(S_{\epsilon/n}) = \infty$.

Note that if $\epsilon'' < \epsilon' < \epsilon_0$, the rectifiability of $S_{\epsilon'}$ implies that $S_{\epsilon''}$ is also rectifiable. This follows from the fact that the nearest-point projection π_Z to a complete, convex subset Z is distance non-increasing in a ball of radius $< \epsilon_0$ in a $\text{CAT}(\kappa)$ space (see, e.g. [BH99, Prop. II.2.4 (or the exercise following for $\kappa > 0$)]). Applying this to the complete convex subset $Z := B(p, \epsilon'')$, and using the (global) $\text{CAT}(\kappa)$ geometry in $B(p, \epsilon_0)$, we see that π_Z is just the radial projection towards p . In particular the image of π_Z lies on $S_{\epsilon''}$, showing that $l(S_{\epsilon''}) \leq l(S_{\epsilon'}) < \infty$.

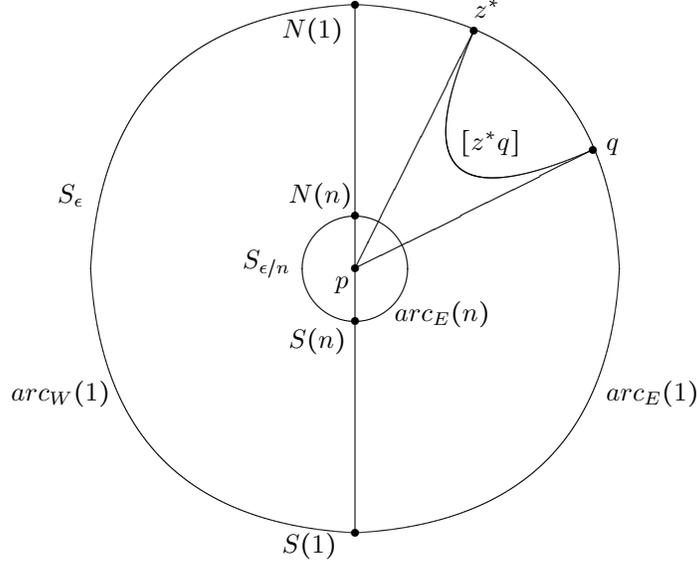


FIGURE 2. A geodesic configuration which cannot occur if $S_{\epsilon/n}$ is not rectifiable.

Fix any geodesic γ through p and denote by $N(n)$ and $S(n)$ its two intersections with $S_{\epsilon/n}$ (chosen so that all $N(n)$ lie on the same component of $\gamma \setminus \{p\}$). Since $S_{\epsilon/n}$ is a circle, the pair $\{N(n), S(n)\}$ divides $S_{\epsilon/n}$ into two arcs, whose closures we call $\text{arc}_E(n)$ and $\text{arc}_W(n)$; choose these so that the relative positions of $N(n), S(n), \text{arc}_E(n)$ and $\text{arc}_W(n)$ correspond to the cardinal directions on a compass.

Note that for all n , at least one of $l(\text{arc}_E(n)), l(\text{arc}_W(n))$ must be infinite, since $l(S_{\epsilon/n})$ is infinite. Without loss of generality, we may assume that $l(\text{arc}_E(n)) = \infty$ for infinitely many n , and hence (by the discussion above) for all n . We focus our attention now on the family $\{\text{arc}_E(n)\}_{n \in \mathbb{N}}$.

On $\text{arc}_E(1)$ define the following equivalence relation: we declare $x \sim y$ if there exists some n such that the arc in $\text{arc}_E(n)$ with endpoints $[px] \cap \text{arc}_E(n)$ and $[py] \cap \text{arc}_E(n)$ is of finite length. That this is an equivalence relation is easy to check. We denote equivalence classes by $[x]$.

We note two things about $\text{arc}_E(1)/\sim$ and its equivalence classes. First, $N(1) \not\sim S(1)$, for otherwise $\text{arc}_E(n)$ would have finite length for some n . Second, for each $x \in \text{arc}_E(1)$, $[x]$ is an interval (possibly degenerate). This is because geodesics are unique at the scale we work at, and if three points are arranged around $\text{arc}_E(1)$ in order $x < y < z$, then $[px] \cap \text{arc}_E(n) \leq [py] \cap \text{arc}_E(n) \leq [pz] \cap \text{arc}_E(n)$. Here we

use $<, \leq$ to denote the ordering of points as they occur along the path $\text{arc}_E(n)$ from $N(n)$ to $S(n)$. Thus the decomposition of $\text{arc}_E(1)$ into the equivalence classes of \sim is a decomposition into at least two disjoint subintervals (possibly degenerate) of the half-circle $\text{arc}_E(1)$.

By connectedness of $\text{arc}_E(1)$, either $[N(1)]$ or $[S(1)]$ is a singleton, or some equivalence class has a closed endpoint in the interior of $\text{arc}_E(1)$. Let z^* be this endpoint or the singleton $N(1)$ or $S(1)$.

If z^* is an endpoint of $\text{arc}_E(1)$, let q be any other point in $\text{arc}_E(1)$. If z^* is the closed endpoint of $[z^*]$ in the interior of $\text{arc}_E(1)$, let q be any point in $\text{arc}_E(1)$ which lies on the z^* -side of $[z^*]$. We note that there are infinitely many such q , and by the choice of z^* and the topology of the half-circle $\text{arc}_E(1)$, we may take a sequence of such q approaching z^* . Observe that, since q and z^* are not equivalent, the geodesic segments $[pz^*]$ and $[pq]$ only agree at the point p .

Consider the geodesic segment $[z^*q]$. By the properties of geodesics in the (globally) $\text{CAT}(\kappa)$ set $B(p, \epsilon)$, this geodesic segment lies inside $B(p, \epsilon)$ and does not cross the geodesic γ which divides the West and East parts of $B(p, \epsilon)$. Suppose that $[z^*q]$ does not intersect $\text{arc}_E(n)$ for some n (as in Figure 2). Then the radial projection of $[z^*q]$ onto S_n provides a path in $\text{arc}_E(n)$ from $[pz^*] \cap \text{arc}_E(n)$ to $[pq] \cap \text{arc}_E(n)$. Again by the distance non-increasing properties of the projection, since $[z^*q]$ has finite length, this would imply $z^* \sim q$, which contradicts the choice of q . Therefore the geodesic segment $[z^*q]$ must intersect $\text{arc}_E(n)$ for all n . It must therefore hit p , and by uniqueness of geodesics we conclude that $[z^*q] = [z^*p] \cup [pq]$.

Now consider $B(z^*, \epsilon)$. The work above shows that no q chosen as previously described lies in $B(z^*, \epsilon)$. But this contradicts our observation above that, using the half-circle topology of $\text{arc}_E(1)$, we may take such q approaching z^* . This contradiction implies that for some n , $S_{\epsilon/n}$ must be rectifiable, concluding the proof. \square

Using the compactness of Σ , we can now promote the pointwise result in Lemma 8 to a global result:

Lemma 9. *Let Σ be a closed $\text{CAT}(\kappa)$ surface. Then there exist $\delta_0 > 0$ and some uniform $C > 0$ such that for all $p \in \Sigma$, $l(S(p, \delta_0)) < C$.*

Proof. Suppose there is no finite, uniform bound on $l(S(p, \delta_0))$, for any δ_0 . Then we may take a sequence of points p_n in Σ with $l(S(p_n, 1/n)) \geq n$. Let p^* be any subsequential limit point of (p_n) . We know by Lemma 8 that there exists some $\epsilon > 0$ such that $l(S(p^*, \epsilon)) < \infty$. For n sufficiently large, $B(p^*, \epsilon)$ properly contains $S(p_n, 1/n)$. But then the unbounded lengths of the latter, plus again the distance non-increasing properties of nearest-point projection, applied to the projection from $S(p^*, \epsilon)$ to $B(p_n, 1/n)$, would imply that the length of $S(p^*, \epsilon)$ is infinite, contradicting Lemma 8. This proves the Lemma. \square

4. PROOF OF THEOREM 1

We define a particular non-expanding map from $B(p, \epsilon_0)$ to the ball of radius ϵ_0 in the model space \mathbb{H}^2 . This will be a key tool in our proof of Theorem 1.

Define an equivalence relation on the set of geodesic segments starting at p by declaring $\gamma_1 \sim \gamma_2$ if the Alexandrov angle between these segments at p is 0.

Definition 10. (See, e.g. [BH99, Definition II.3.18]) The set of equivalence classes for \sim , equipped with the metric provided by the Alexandrov angle, is the *space of directions at p* , denoted $S_p(\Sigma)$.

The following result is standard:

Proposition 11. (See, e.g. [BH99, Theorem II.3.19]) *If $\tilde{\Sigma}$ is $CAT(\kappa)$ for any κ , then for each $p \in \tilde{\Sigma}$, the completion of $S_p(\tilde{\Sigma})$ is $CAT(1)$.*

Lemma 12. *For any $p \in \tilde{\Sigma}$, where $\tilde{\Sigma}$ is a $CAT(\kappa)$ surface, $S_p(\tilde{\Sigma}) \cong \mathbb{S}^1$.*

Proof. The natural projection from S_ϵ to $S_p(\tilde{\Sigma})$ is continuous and, by Lemma 4, surjective. The fiber over any point in $S_p(\tilde{\Sigma})$ is easily seen to be a closed interval. Thus $S_p(\tilde{\Sigma})$ is homeomorphic to a quotient of \mathbb{S}^1 , where each equivalence class is a closed interval in \mathbb{S}^1 . It is a well-known result that such a quotient space is automatically homeomorphic to \mathbb{S}^1 (this can be easily shown using the topological characterization of \mathbb{S}^1 used in the proof of Proposition 5). This establishes the Lemma. \square

Combining Proposition 11 with Lemma 12, which implies that $S_p(\tilde{\Sigma})$ is complete, we have

Corollary 13. *For any $p \in \tilde{\Sigma}$, $S_p(\tilde{\Sigma})$ is $CAT(1)$.*

We now construct the non-expanding map to the model surface M_κ of constant curvature κ . We closely follow the proof of a similar result presented in [BBI01, Proposition 10.6.10], but for the opposite type of curvature bound (curvature bounded below, rather than above).

Proposition 14. *Let Σ be a $CAT(\kappa)$ surface and p any point in Σ . Let ϵ_0 be as above. Then there is a map $f : B(p, \epsilon_0) \rightarrow M_\kappa$ such that*

- (1) $d_{M_\kappa}(f(x), f(y)) \leq d_\Sigma(x, y)$ for all $x, y \in B(p, \epsilon_0)$,
- (2) $d_{M_\kappa}(f(p), f(y)) = d_\Sigma(p, y)$ for all $y \in B(p, \epsilon_0)$, and
- (3) $f(B(p, \epsilon)) = B_{M_\kappa}(f(p), \epsilon)$.

Proof. By the choice of ϵ_0 , we can work in Σ or lift $B(p, \epsilon_0)$ homeomorphically to $\tilde{\Sigma}$. By corollary 13, $S_p(\tilde{\Sigma})$ is $CAT(1)$. It is homeomorphic to \mathbb{S}^1 , so it is easy to see that there is a surjective map $g : S_p(\tilde{\Sigma}) \rightarrow \mathbb{S}^1$ which is non-expanding:

$$d_{\mathbb{S}^1}(g(v), g(w)) \leq \angle_p(v, w) \quad \text{for all } v, w \in S_p(\tilde{\Sigma}).$$

Let $K_p^\kappa(\tilde{\Sigma})$ denote the κ -cone over $S_p(\tilde{\Sigma})$. This space is topologically a cone over $S_p(\tilde{\Sigma})$ with origin denoted o and coordinates (v, r) away from o , where $v \in S_p(\tilde{\Sigma})$ and $r > 0$ (with r truncated at $\pi/\sqrt{\kappa}$ if $\kappa > 0$). It is equipped with a metric devised so that the κ -cone over the circle is the model space of curvature κ . For details of its construction see, e.g., [BBI01, §10.2.1]. Its key property for our purposes is the following: since $S_p(\tilde{\Sigma})$ is $CAT(1)$, $K_p^\kappa(\tilde{\Sigma})$ is $CAT(\kappa)$ ([BH99, Theorem II.3.14]). On the ball $B(p, \epsilon_0)$ define a logarithm map as follows:

$$\begin{aligned} \log_p : B(p, \epsilon_0) &\rightarrow K_p^\kappa(\tilde{\Sigma}) \\ p &\mapsto o \\ x \neq p &\mapsto (v, d_\Sigma(p, x)) \end{aligned}$$

where v is the direction in $S_p(\tilde{\Sigma})$ of the geodesic segment $[px]$. From the non-expanding property of g and the definition of $K_p^\kappa(\tilde{\Sigma})$,

$$d_{K_p^\kappa(\tilde{\Sigma})}(\log_p(x), \log_p(y)) \leq d_\Sigma(x, y) \quad \text{for all } x, y \in B(p, \epsilon_0).$$

By its definition, \log_p preserves distance from the origin, and by its definition, \log_p maps $B(p, \epsilon)$ surjectively onto $B_{K_p^{-1}(\tilde{\Sigma})}(o, \epsilon)$.

Again, using the definition of $K_p^\kappa(\tilde{\Sigma})$, the non-expanding map $g : S_p(\tilde{\Sigma}) \rightarrow \mathbb{S}^1$ extends to a map $G : K_p^\kappa(\tilde{\Sigma}) \rightarrow M_\kappa$, obtained by realizing M_κ as the κ -cone over \mathbb{S}^1 . The map is non-expanding since g is, and preserves distance from the origin. G sends $B_{K_p^\kappa(\tilde{\Sigma})}(o, \epsilon)$ surjectively to $B_{M_\kappa}(G(o), \epsilon)$ because g is surjective. Then $f = G \circ \log_p$ is the desired map. \square

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let $\delta_0 < \epsilon_0$ be given by Lemma 9 and let $\delta < \delta_0$. First we bound $H^2(B(p, \delta))$ below. Recall that the Hausdorff 2-measure of some metric space X is defined via a two step process. For some $\rho > 0$, one considers open covers $\{U_i\}$ of X by open sets of diameter $< \rho$, and takes the infimum of $\sum \text{diam } U_i^2$ over all such covers. This defines the quantity $H_\rho^2(X)$, which is non-increasing as a function of ρ . The Hausdorff 2-measure is then the supremum of the $H_\rho^2(X)$ (which of course coincides with the limit of these as $\rho \rightarrow 0$).

Let f be the non-expanding map provided by Proposition 14. Since f preserves radial distance from the origin, $f(B(p, \delta)) \subseteq B(f(p), \delta)$. Fix any $\rho > 0$ and suppose $\{U_i\}$ is a countable cover of $B(p, \delta)$ with $\text{diam}(U_i) < \rho$. Then the collection $\{f(U_i)\}$ covers $B(f(p), \delta)$ and $\text{diam}(f(U_i)) \leq \text{diam}(U_i) < \rho$ as f is non-expanding. Therefore,

$$\sum_i \text{diam } U_i^2 \geq \sum_i \text{diam } f(U_i)^2 \geq H_\rho^2(B(f(p), \delta)).$$

Passing to the infimum over all such covers $\{U_i\}$ of $B(p, \delta)$, we obtain for each $\rho > 0$ the inequality $H_\rho^2(B(p, \delta)) \geq H_\rho^2(B(f(p), \delta))$. Passing to the limit as $\rho \rightarrow 0$, this gives $H^2(B(p, \delta)) \geq H^2(B(f(p), \delta))$. Finally, one observes that the right hand side $H^2(B(f(p), \delta))$ is just the volume of a δ ball in the model space M_κ . Since this quantity is independent of the choice of point p , this gives the desired uniform lower bound on $H^2(B(p, \delta))$.

Now we bound $H^2(B(p, \delta))$ above. This portion of the proof uses the uniform bound on the length of S_δ obtained in Lemma 9. It is sufficient to bound $H^2(B(p, \delta_0))$ uniformly above.

Fix $\rho < \delta_0$. Let E_ρ be any minimal cardinality subset of S_{δ_0} which is $\frac{\rho}{2}$ -dense inside S_{δ_0} . The circumference bound allows us to uniformly bound $\#E_\rho$. Index $x_j \in E_\rho$ in order around S_{δ_0} . Let T_j be the geodesic triangle with vertices p, x_j, x_{j+1} . T_j has edges of length δ_0, δ_0 and $< \rho$. Let τ_j be the corresponding comparison triangle in M_κ (see Figure 3).

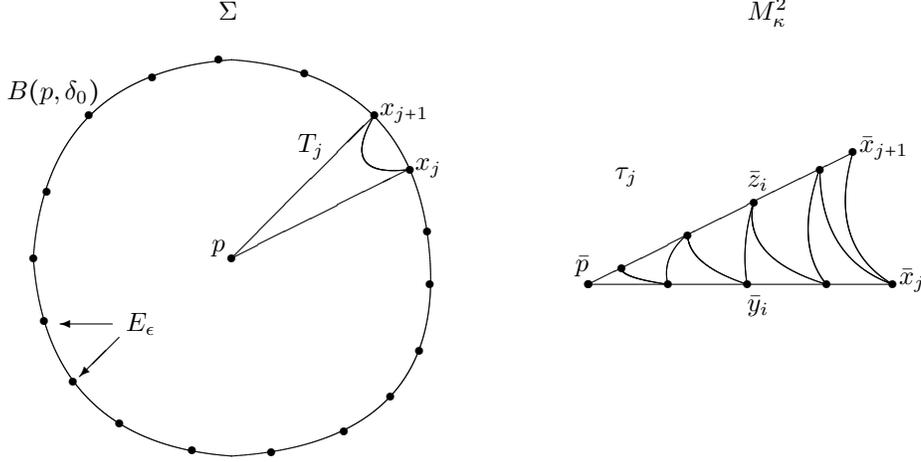


FIGURE 3. Comparison triangles for the upper bound.

In M_κ , let r_j be the δ_0 -length edge from \bar{p} to \bar{x}_j . Let $r(\rho)$ be the number of ρ -balls centered at points on r_j necessary to cover τ_j with centers an ρ -spanning set in r_j . Let their centers be $\bar{y}_1, \dots, \bar{y}_{r(\rho)}$. Note that we can take $r(\rho) = C' \frac{\delta_0}{\rho}$ for C' a constant independent of ρ . Now, pick \bar{z}_i on the other δ_0 -length side of τ_j and in $B(\bar{y}_i, \rho) \cap B(\bar{y}_{i+1}, \rho)$. Draw in τ_j the zig-zagging geodesic segments connecting \bar{y}_i to \bar{z}_i to \bar{y}_{i+1} to \bar{z}_{i+1} etc. These partition τ_j into a union of triangles. In τ_j each small triangle in the subdivision has all three sides of length $< \rho$, by the choice of \bar{y}_i and \bar{z}_i . Using the comparison triangles, pick corresponding points y_i, z_i in T_j , and form the corresponding zig-zagging segments in T_j , obtaining a subdivision of T_j into a union of triangles. By the $\text{CAT}(\kappa)$ condition, the corresponding small triangles in T_j also have all sides of length $< \rho$, and then, again by $\text{CAT}(\kappa)$, we see that the ρ -balls centered at y_i in T_j cover T_j .

Since $l_g(S_{\delta_0}) < C$, we can take $\#E_\rho \leq \frac{2C}{\rho}$. Then $B(p, \delta_0)$ can be covered by $2CC' \frac{\delta_0}{\rho^2}$ balls of radius ρ . Thus we obtain a finite cover $\{U_i\}$ of $B(p, \delta_0)$ with $\text{diam}(U_i) \leq 2\rho$ satisfying,

$$\sum_i \text{diam}(U_i)^2 \leq 2CC' \frac{\delta_0}{\rho^2} (2\rho)^2 = 8CC' \delta_0$$

which is bounded above independently of ρ . It follows that $H^2(B(p, \delta_0)) \leq 8CC' \delta_0$, which verifies that $H^2(B(p, \delta_0))$ is finite and bounded above uniformly in p . \square

5. THE TOPOLOGY OF SMALL DISTANCE SPHERES IN HIGHER DIMENSIONS

In this section we make a few remarks on the obstructions to extending the local geometry results we proved for surfaces to higher dimensions. We note that the proofs in the previous section rely heavily on the 2-dimensionality of Σ . We do not know if an analogue of Theorem 1 holds for $\text{CAT}(\kappa)$ metrics on closed higher

dimensional manifolds. One of the first steps in our proof was Proposition 5, which showed that the small enough metric spheres inside locally $\text{CAT}(\kappa)$ surfaces were homeomorphic to the circle \mathbb{S}^1 . The analogous statement *fails* in dimensions ≥ 5 , as the well-known example below shows.

Proposition 15 (Davis-Januszkiewicz). *For each dimension $n \geq 5$, there exists a closed n -manifold M equipped with a piecewise hyperbolic, locally $\text{CAT}(-1)$ metric, and a point $p \in M$ with the property that for all small enough ϵ , the ϵ -sphere S_ϵ centered at p is **not** homeomorphic to \mathbb{S}^{n-1} . In fact, S_ϵ is not even a manifold.*

Proof. Such examples can be found in the work of Davis and Januszkiewicz [DJ91, Theorem 5b.1]. We briefly summarize the construction for the convenience of the reader. Start with a closed smooth homology sphere N^{n-2} which is not homeomorphic to \mathbb{S}^{n-2} . Such manifolds exist for all $n \geq 5$, and are quotients of \mathbb{S}^{n-2} by a suitable perfect group $\pi_1(N^{n-2})$. Take a smooth triangulation of N^{n-2} , and consider the induced triangulation \mathcal{T} on the double suspension $\Sigma^2(N^{n-2})$. By work of Cannon and Edwards [Can79, Edw], $\Sigma^2(N^{n-2})$ is homeomorphic to \mathbb{S}^n . The triangulation \mathcal{T} on \mathbb{S}^n is not a PL-triangulation, as there exists a 4-cycle in the 1-skeleton of the triangulation whose link is homeomorphic to N^{n-2} . Now apply the strict hyperbolization procedure of Charney and Davis [CD95] to the triangulated manifold $(\mathbb{S}^n, \mathcal{T})$.

This outputs a piecewise-hyperbolic, locally $\text{CAT}(-1)$ space M . A key point of the hyperbolization procedure is that it preserves the local structure. Since the input $(\mathbb{S}^n, \mathcal{T})$ is a closed n -manifold, the output M is also a closed n -manifold. The 4-cycle in \mathcal{T} whose link was homeomorphic to N^{n-2} now produces a closed geodesic γ in M , whose link is still homeomorphic to N^{n-2} (i.e. the “unit normal” to γ forms a copy of N^{n-2}). It follows from this that, picking the point p on γ , all small ϵ -spheres S_ϵ are homeomorphic to the suspension ΣN^{n-2} . Since N^{n-2} was **not** the standard sphere, $S_\epsilon \cong \Sigma N^{n-2}$ fails to be a manifold at the suspension point x , as every small punctured neighborhood of x will have non-trivial π_1 . We refer the reader to [DJ91, Section 5] for more details. □

This cautionary example suggests that small metric spheres in high-dimensional locally $\text{CAT}(\kappa)$ manifolds could exhibit pathologies. In view of these results, and the interest in obtaining higher dimensional analogs, we raise the following question.

Question . Let M be a closed n -manifold equipped with a locally $\text{CAT}(-1)$ metric of Hausdorff dimension d . Can d ever be strictly larger than n ? Do the uniform bound conditions of Theorem 1 hold in higher dimensions?

The authors suspect that examples with $d > n$ do indeed exist in higher dimensions.

6. ENTROPY RIGIDITY IN $\text{CAT}(-1)$

In this section we present an entropy rigidity result for closed $\text{CAT}(-1)$ manifolds. This result generalizes Hamenstädt’s entropy rigidity result from [Ham90] to the $\text{CAT}(-1)$ setting. It is very closely related to, and in fact relies on, a rigidity result of Bourdon. The main addition to Bourdon’s theorem is the connection to topological entropy via a theorem of Leuzinger (generalizing work of Manning).

The theorem draws heavily on the work of others, but we have not seen it presented in this form in the literature.

We remark that our work on Hausdorff 2-measure for surfaces, presented in the earlier sections of this paper, was inspired in part by condition (\star) that features in Leuzinger's theorem below (Theorem 21).

Recall that a negatively curved locally symmetric space has universal cover isometric to $\mathbb{H}_{\mathbf{K}}^m$, where $m \geq 2$ and \mathbf{K} is \mathbb{R}, \mathbb{C} , the quaternions \mathbf{H} , or the octonions \mathbf{O} (with $m = 2$). We suppose that the metrics on these spaces are scaled so that the maximum sectional curvature is -1 . The topological entropy for the geodesic flow on a compact locally symmetric space modeled on $\mathbb{H}_{\mathbf{K}}^m$ is $km + k - 2$ where $k = \dim_{\mathbb{R}} \mathbf{K}$. The topological entropy measures the exponential growth rate of the number of orbits that are distinguishable over longer and longer periods of time (for a precise definition of topological entropy, see [KH95, §3.1].)

More generally, if one has a locally $\text{CAT}(-1)$ metric d on the manifold M , there is an associated space $\mathcal{G}(M, d)$ of geodesics in M : this is the space of locally isometric maps $\mathbb{R} \rightarrow M$. Via lifting to the universal cover, this space is topologically a quotient of $\partial^\infty \tilde{M} \times \partial^\infty \tilde{M} \times \mathbb{R}$ by a suitable $\Gamma := \pi_1(M)$ action. There is a natural flow ϕ_t^d on $\mathcal{G}(M, d)$, given by precomposition with an \mathbb{R} -translation. This is called the geodesic flow associated to the metric d . One can again measure the topological entropy of this flow.

The main result of this section is the following restatement of our Theorem 3:

Theorem 16. *Let (X, d) be a closed n -dimensional manifold equipped with a locally $\text{CAT}(-1)$ metric d (not necessarily Riemannian). Suppose that X also supports a locally symmetric Riemannian metric g , under which (X, g) is locally modeled on $\mathbb{H}_{\mathbf{K}}^m$, normalized to have maximum sectional curvature -1 . Then*

$$h_{\text{top}}(\phi_t^d) \geq h_{\text{top}}(\phi_t^g) = km + k - 2$$

and if equality holds in the above, (X, d) is also locally symmetric. If $n > 2$, (X, d) and (X, g) are isometric.

This should be compared with the main theorem of [Ham90], which establishes this same result when the metric d is a Riemannian metric with sectional curvature ≤ -1 . The key element in this proof is the following theorem of Bourdon, which he notes is a generalization of Hamenstädt's work.

Theorem 17. [Bou95, Théorème 0.3 and following remarks] *Let \tilde{X} be a $\text{CAT}(-1)$ space with a cocompact isometric action by Γ which also acts convex cocompactly by isometries on a negatively curved symmetric space $S = \mathbb{H}_{\mathbf{K}}^m$. Let Λ be the limit set in $\partial^\infty \tilde{X}$ of Γ , and let d^* and g^* denote the visual metrics on $\partial^\infty \tilde{X}$ and $\partial^\infty S$, respectively. Suppose that $\dim_H(\Lambda, d^*) = \dim_H(\partial^\infty S, g^*)$. Then there exists an isometric embedding F from S into \tilde{X} such that $\partial^\infty F(\partial^\infty S) = \Lambda$. If $\dim_{\mathbb{R}} S > 2$, then this embedding is Γ -equivariant.*

This theorem proves rigidity for the extremal value of $\dim_H(\Lambda, d^*)$ – an extremal value which had already been computed by Pansu:

Theorem 18. [Pan89, Théorème 5.5] *With notation as in Theorem 17,*

$$\dim_H(\Lambda, d^*) \geq \dim_H(\partial^\infty S, g^*) = km + k - 2.$$

A definition for the visual metrics referenced above can be found in [BH99, §III.H.3]. Here we give a short definition of a distance function which is Lipschitz

equivalent to any visual metric. Since Hausdorff dimension can be calculated using any distance function in the Lipschitz equivalence class, this suffices for our purposes.

Definition 19. Let (\tilde{X}, \tilde{d}) be a CAT(-1) metric space and let $\zeta, \eta \in \partial^\infty \tilde{X}$. Fix some basepoint $x \in \tilde{X}$ and define

$$d_x^*(\zeta, \eta) = e^{-\tilde{d}(x, [\zeta\eta])},$$

where $[\zeta\eta]$ denotes the bi-infinite geodesic in \tilde{X} with endpoints at ζ and η . It is straightforward to see that the Lipschitz class of d_x^* is independent of x .

We also need the following result of Manning, as generalized to the CAT(-1) setting by Leuzinger.

Definition 20. Let (X, d) be a closed manifold with some metric d and endowed with a measure vol . Let (\tilde{X}, \tilde{d}) denote its universal cover. Then the *volume growth entropy* of (\tilde{X}, \tilde{d}) with respect to vol is

$$h_{vol}(\tilde{X}, \tilde{d}) := \limsup_{R \rightarrow \infty} \frac{1}{R} vol(B(p, R))$$

where $B(p, R)$ is the ball of radius R about a point p in \tilde{X} .

Manning shows that this is independent of the choice of p and that, for Riemannian manifolds, the limsup is in fact a limit.

Theorem 21 (Leuzinger, [Leu06]; compare with Manning [Man79]). *Let (\tilde{X}, \tilde{d}) be a geodesically complete, locally geodesic metric space endowed with a measure vol , and having compact quotient \tilde{X}/Γ . Assume that*

$$(\star) \text{ there exists some } 0 < \delta_0 < \infty \text{ such that for all } 0 < \delta \leq \delta_0, \\ 0 < \inf_{z \in \tilde{X}} vol(B(z, \delta)) \text{ and } \sup_{z \in \tilde{X}} vol(B(z, \delta)) < \infty.$$

Then

$$h_{top}(\phi_t) \geq h_{vol}(\tilde{X}, \tilde{d}).$$

If, in addition, (\tilde{X}, \tilde{d}) is CAT(0), then

$$h_{top}(\phi_t) = h_{vol}(\tilde{X}, \tilde{d}).$$

Manning proved this result for Riemannian manifolds (for which condition (\star) holds automatically). Leuzinger extracts the key conditions from that proof, namely condition (\star) as well as convexity of the function $d(c_1(t), c_2(t))$, the distance between a pair of geodesics. The convexity condition is automatically satisfied in a CAT(0) space – which just leaves condition (\star) to verify.

In the surface case, the validity of condition (\star) for 2-dimensional Hausdorff measure is established in our Theorem 1, so applying Theorem 21 immediately yields Corollary 2.

Finally, we note the following.

Proposition 22. *Let (X, d) be compact and locally CAT(-1), and suppose that vol is a measure on X giving finite, nonzero measure to X . Then*

$$h_{vol}(\tilde{X}, \tilde{d}) = \dim_H(\partial^\infty \tilde{X}, d_x^*)$$

where \dim_H denotes Hausdorff dimension.

Proof of Prop 22. Let $N(R) = \#\{\gamma \in \Gamma : d_{\tilde{X}}(*, \gamma*) \leq R\}$. The result follows easily from the following facts. First, due to Bourdon [Bou95, Theorem 2.7.5], we have that

$$\dim_H(\partial^\infty \tilde{X}, d_x^*) = \limsup_{R \rightarrow \infty} \frac{1}{R} \log N(R).$$

Second, if we let F be a bounded fundamental domain for the Γ -action, with volume V and diameter D ,

$$\text{vol}(B(*, R)) \leq V \cdot N(R + D)$$

and

$$V \cdot N(R) \leq \text{vol}(B(*, R + D)).$$

□

We now prove Theorem 16.

Proof of Theorem 16. Since X is a smooth manifold, we may fix some Riemannian metric on X , for example the locally symmetric metric g (though any metric will do). This defines a Riemannian volume form vol on X . We first claim that vol satisfies condition $(*)$ from Theorem 21.

Indeed, take δ_0 so small that any δ_0 -ball for the d -metric on X lifts isometrically to (\tilde{X}, \tilde{d}) . Then the volume of such a ball is clearly bounded above by the Riemannian volume of X , which is finite as X is compact. This establishes the uniform upper bound on $\text{vol}(B^d(z, \delta))$.

Suppose that there is no uniform positive lower bound. Then for any fixed $\delta > 0$, there exists a sequence of points z_n in X such that $\text{vol}(B^d(z_n, \delta)) \rightarrow 0$. By compactness we may extract a convergent subsequence z_{n_i} with limit z^* . For sufficiently large i , $B^d(z^*, \delta/2) \subset B^d(z_{n_i}, \delta)$ and hence $\text{vol}(B^d(z^*, \delta/2)) = 0$. Since d and the Riemannian metric g induce the same topology on X , there exists some $\epsilon > 0$ such that $B^g(z^*, \epsilon) \subset B^d(z^*, \delta/2)$. But the Riemannian volume of $B^g(z^*, \epsilon)$ must be strictly positive, giving a contradiction. Therefore, condition $(*)$ holds for vol .

Let (S, g) be the negatively curved symmetric space on which (X, g) is locally modelled. Using Theorem 21, Proposition 22 and Theorem 18 and the fact that Γ acts cocompactly on X (and so $\Lambda = \partial^\infty \tilde{X}$), we have

$$\begin{aligned} h_{\text{top}}(\phi_t^d) &= h_{\text{vol}}(\tilde{X}, \tilde{d}) \\ &= \dim_H(\partial^\infty \tilde{X}, d_x^*) \\ &\geq \dim_H(\partial^\infty S, g_x^*) \\ &= kn' + k - 2. \end{aligned}$$

Now suppose that $h_{\text{top}}(\phi_t^d)$ achieves the lower bound. Then by Theorem 17, there exists an isometric embedding F from S into X with $\partial^\infty F(\partial^\infty S) = \Lambda = \partial^\infty \tilde{X}$; if $n > 2$, this embedding can be taken to be Γ -equivariant. Since X is a manifold with $\dim(X) = \dim(S)$ and $\partial^\infty F(\partial^\infty S)$ is the full boundary at infinity of \tilde{X} , it follows that this isometric embedding is in fact surjective. This finishes the proof. □

Remark. If instead of a smooth manifold, we just had a *topological* manifold M equipped with a locally CAT(-1) metric, then one cannot use a Riemannian volume form to obtain a measure satisfying condition $(*)$. In this setting, the Hausdorff measure seems to be the only natural measure to use. It is unclear to the authors

whether or not the Hausdorff measure must satisfy the regularity condition (*). Note that Davis and Januszkiewicz [DJ91] give examples, in dimensions $n \geq 5$, of non-smoothable closed topological manifolds M^n which support locally $CAT(-1)$ metrics.

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