Unique Equilibrium States for Geodesic Flows on Flat Surfaces with Singularities

Benjamin Call[1], David Constantine[2], Alena Erchenko[3,*], Noelle Sawyer[4], and Grace Work[5]

[1]Department of Mathematics, Statistics, and Computer Science, University of Illinois Chicago, Chicago, IL 60607, USA, [2]Mathematics and Computer Science Department, Wesleyan University, Middletown, CT 06459, USA, [3]Department of Mathematics, The University of Chicago, Chicago, IL 60637, USA, [4]Mathematics and Computer Science Department, Southwestern University, Georgetown, TX 78626, USA, and [5]Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA

Correspondence to be sent to: e-mail: aerchenko@uchicago.edu

Consider a compact surface of genus ≥ 2 equipped with a metric that is flat everywhere except at finitely many cone points with angles greater than 2π. Following the technique in the work of Burns, Climenhaga, Fisher, and Thompson, we prove that sufficiently regular potential functions have unique equilibrium states if the singular set does not support the full pressure. Moreover, we show that the pressure gap holds for any potential that is locally constant on a neighborhood of the singular set. Finally, we establish that the corresponding equilibrium states have the K-property and closed regular geodesics equidistribute.

1   Introduction

We examine the uniqueness of equilibrium states for geodesic flows on a specific class of CAT(0) surfaces, those where the negative curvature is concentrated at a finite set of

points. Translation surfaces are examples of such surfaces. A translation surface X is a pair (X, ω) where X is a Riemann surface of genus g, and ω is a holomorphic one-form on X. The zeroes of this holomorphic one-form occur at a finite set of points. The one-form ω defines a metric that is flat everywhere except at its zeroes. At the zeroes, the metric has a conical singularity with angle $2(n + 1)\pi$, where n is the order of the zero. For a more in-depth overview of translation surfaces, see [22, 23].

In [4], the authors prove that under certain conditions, a unique equilibrium state exists for potentials associated with the geodesic flow on a closed, rank-one manifold with nonpositive sectional curvature (an example of a CAT(0) space *without* singularities). The conditions are a Hölder continuous potential and a pressure gap, that is, topological pressure of the flow restricted to the singular set is strictly less than pressure of the flow overall. The singular set they consider is all the vectors in the unit tangent bundle with rank larger than one.

When the singular set is empty—for example, in strictly negative curvature—every Hölder potential has a unique equilibrium state. When the singular set is non-empty, an additional condition is necessary as the geodesic flow is nonuniformly hyperbolic. Restricting the pressure of the flow on the singular set is a way of describing the flow of the singular set as having a small enough impact on the system as a whole that uniqueness is still guaranteed.

The natural way to define a geodesic flow on CAT(0) surfaces is to look at the flow on the set of all geodesics (see Section 2.1). Denote by GS the set of all geodesics on the surface S (see (1)).

In this paper, we study the uniqueness of equilibrium states for the geodesic flow described above (see Definition 2.5), as we are guaranteed existence for continuous potentials by entropy-expansivity of the flow (see Lemma 2.17). In particular, we use the technique of [4] in our setting and define the singular set to be the set of geodesics that never encounter any cone points or, when they do, turn by angle exactly $\pm\pi$.

Remark.   Some other settings where the uniqueness of equilibrium states were studied are described in more detail below in the outline of the argument.

We prove the following.

Theorem A.   Let g_t be the geodesic flow on S, a compact, connected surface of genus ≥ 2 equipped with a metric that is flat everywhere except at finitely many cone points that have angle greater than 2π. Let Sing be the singular set as defined in Definition 2.4.

Consider $\phi\colon GS \to \mathbb{R}$ a Hölder continuous potential. If the pressure of the singular set is strictly less than the full topological pressure, that is, $P(\mathrm{Sing}, \phi) < P(\phi)$ (see Definitions 2.5 and 2.6), then ϕ has a unique equilibrium state μ that has the K-property (see Definition 2.2).

It is natural to ask for which potentials we have the *pressure gap* (i.e., the condition $P(\mathrm{Sing}, \phi) < P(\phi)$) in Theorem A. The following theorem establishes the pressure gap for a large class of Hölder continuous potentials and thus uniqueness of equilibrium states.

Theorem B   (Theorem 7.1 and Corollary 7.8).   Let S, GS, and g_t be as in Theorem A. Let $\phi\colon GS \to \mathbb{R}$ be a Hölder continuous function that is locally constant on a neighborhood of Sing, or which is sufficiently close to a constant in the uniform norm (see Corollary 7.8 for a precise statement of "sufficiently close"). Then $P(\mathrm{Sing}, \phi) < P(\phi)$.

As a nice corollary (Corollary 7.7 below), we have $h_{top}(g_t|_{\mathrm{Sing}}) < h_{top}(g_t)$ for our flows.

We slightly improve the case $\phi = 0$ from Ricks's result [20, Theorem B] by showing that the unique measure of maximal entropy for the geodesic flow on S has the K-property that is stronger than mixing. Using the Patterson–Sullivan construction, Ricks builds a measure of maximal entropy μ [19] and shows it is unique by asymptotic geometry arguments [20]. We note that Ricks's result holds for any compact, geodesically complete, locally CAT(0) space such that the universal cover admits a rank-one axis.

A natural question is whether the techniques in this paper can be extended to the more general CAT(0), rank-one setting in which Ricks works. The present paper can be viewed as a 1st step in that direction, but working in the general CAT(0) setting presents real difficulties right from the outset of the argument. In particular, without the Riemannian structure present in [4] or the flat surface structure we exploit, it is not clear to us what the right candidate for the singular set for would be or how to find a function like λ (see Section 3) to aid in producing an orbit decomposition.

We call a geodesic that is not in Sing *regular*. Using strong specification for a certain collection of "good" orbit segments, we show that weighted regular closed geodesics equidistribute to these equilibrium states (see Section 8 for details).

Theorem C   (Theorem 8.1).   Let ϕ be as in Theorem B and μ_ϕ is the corresponding equilibrium state. Then, μ_ϕ is the weak* limit of weighted regular closed geodesics.

1.1   Outline of the argument

A general scheme for proving that unique equilibrium states exist was developed by Climenhaga and Thompson [10], building on ideas of Bowen [2] that were extended to flows in [17]. To prove that there are unique equilibrium states for a flow $\{f_t\}$ and a potential ϕ on a compact metric space X, Climenhaga and Thompson ask for the following (see [10, Theorems A and C]).

- The pressure of obstructions to expansivity, $P_{exp}^{\perp}(\phi)$ (see Definition 2.7), is smaller than $P(\phi)$.

- There are three collections of orbit segments $\mathcal{P}, \mathcal{G}, \mathcal{S}$, that we call collections of prefixes, good orbit segments, and suffixes, respectively, such that each orbit segment can be decomposed into a prefix, a good part, and a suffix (see [4, Definition 2.3]), satisfying

  (I) $\mathcal{G}$ has the weak specification property at any scale (Definition 2.8);
  (II) ϕ has the Bowen property on $\mathcal{G}$ (Definition 2.9); and
  (III) $P([\mathcal{P}] \cup [\mathcal{S}], \phi) < P(\phi)$.

This scheme was implemented for the geodesic flow on a closed rank-one manifold with nonpositive sectional curvature in [4] and, more generally, without focal points in [7, 8]. Also, it was used to obtain the uniqueness of the measure of maximal entropy on certain manifolds without conjugate points in [9] and on CAT(-1) spaces in [13].

Our proof follows a specific approach to satisfying the conditions in the above scheme that was applied in [4] and that allows us to reduce condition (III) to checking the pressure of an invariant subset of GS. Although the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ is in general very abstract, we choose the decomposition using a function λ on the space of geodesics. This choice of decomposition also allows us to avoid having to deal with the sets $[\mathcal{P}]$ and $[\mathcal{S}]$, which are discretized versions of $\mathcal{P}$ and S necessary for technical counting arguments to be applied to some decompositions. We define the function λ, prove that it is lower semicontinuous, and describe how it gives rise to a decomposition in Section 3. For such a "λ-decomposition", $\mathcal{P} = \mathcal{S}$ and, roughly speaking, orbit segments in $\mathcal{P}$ and $\mathcal{S}$ have small average values of λ whereas any initial or terminal segment of an element of $\mathcal{G}$ has average value of λ, which is not small. Furthermore, by utilizing a λ-decomposition, we are able to appeal to the following result.

Theorem 1.1. ([6, Theorem 4.6])   Let $\mathcal{F}$ be a continuous flow on a compact metric space X, and let $\phi : X \rightarrow \mathbb{R}$ be continuous. Suppose the flow is asymptotically

entropy-expansive, that $P_{\exp}^{\perp}(\phi) < P(\phi)$, and that $\lambda : X \to [0, \infty)$ is lower semicontinuous and bounded. If the λ-decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ satisfies the following:

- $\mathcal{G}(\eta)$ has strong specification at all scales, for all $\eta > 0$;
- ϕ has the Bowen property on $\mathcal{G}(\eta)$;
- $P(\bigcap_{t \in \mathbb{R}} (f_t \times f_t) \tilde{\lambda}^{-1}(0), \Phi) < 2P(\phi)$,

where $\Phi(x, y) = \phi(x) + \phi(y)$ and $\tilde{\lambda}(x, y) = \lambda(x)\lambda(y)$, then $(X, \mathcal{F}, \phi)$ has a unique equilibrium state which has the K-property.

Theorem A will follow from Theorem 1.1 after we show that we can satisfy all conditions required. See Section 1.2 for the sections where each property is checked.

Our choice of λ gives a connection between orbit segments in $\mathcal{P}$ and $\mathcal{S}$ and the singular set Sing (see Definition 2.4). The singular set is also the source of the obstructions to expansivity (see Lemma 2.16). These connections are useful for proving the two "pressure gap" properties Theorem 1.1 calls for $P_{exp}^{\perp}(\phi) < P(\phi)$ and $P(\bigcap_{t \in \mathbb{R}} (f_t \times f_t) \tilde{\lambda}^{-1}(0), \Phi) < 2P(\phi)$. In particular, in our case, $\bigcap_{t \in \mathbb{R}} f_t \lambda^{-1}(0) = $ Sing.

Remark.   The strong specification property on $\mathcal{G}$ in Theorem 1.1 is used to obtain that the equilibrium state has the K-property. The weak specification property on $\mathcal{G}$ is enough to guarantee the existence of a unique equilibrium state.

Remark.   The K-property implies strong mixing of all orders.

1.2   Organization of the paper

The paper is organized as follows. In Section 2, we provide definitions of and background on the main objects and tools of this paper and we record some basic geometric results that will be used throughout the paper. The main steps for the proof of Theorem A according to Theorem 1.1 are in Sections 3 (the λ-decomposition), 4 and 5 (the specification property for $\mathcal{G}$), and 6 (the Bowen property for $\mathcal{G}$).

We obtain Theorem B in Section 7, first proving the pressure gap condition for potentials that are locally constant on a neighborhood of Sing, and then using this result to note that the same gap holds for potentials with sufficiently small total variation. Theorem C (the equidistribution result) is proved in Section 8.
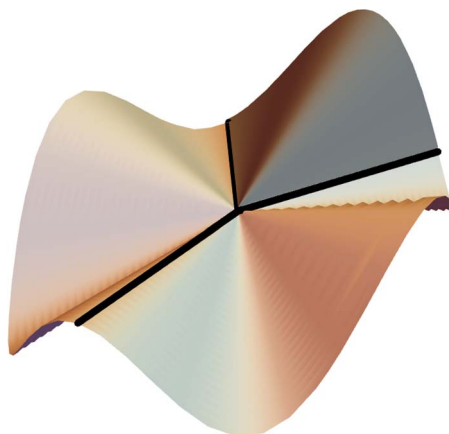
Fig. 1. A large-angle cone point, embedded in $\mathbb{R}^3$. Away from the cone point, the surface is flat under the intrinsic metric—it is the union of lines in $\mathbb{R}^3$ and so has Gaussian curvature zero. The dark lines show a geodesic segment hitting the cone point and its two continuations with turning angles $\pm\pi$; these geodesics are in Sing. All continuations of the geodesic with line segments passing through the dark shaded region are geodesics. The spread of the geodesic continuations in this region is exactly the source of "hyperbolicity" for the geodesic flow in these spaces.

2 Background

2.1 Setting and definitions

Throughout, S denotes a compact, connected surface of genus ≥ 2 equipped with a metric that is flat everywhere except at finitely many conical points that have angles larger than 2π (see Figure 1). We assume S is oriented by passing to the oriented double cover if necessary. *Con* denotes the set of conical points on S and denote by $\mathcal{L}(p)$ the total angle at a point $p \in S$. In particular, $\mathcal{L}(p) = 2\pi$ if $p \notin Con$ and $\mathcal{L}(p) > 2\pi$ if $p \in Con$. Note that in the special case of a translation surface, $\mathcal{L}(p)$ is always an integer multiple of 2π, but we make no such restriction here. Denote by $\tilde{S}$ the universal cover of S, and note that $\tilde{S}$ is a complete CAT(0) space (see, e.g., [3] for definitions and basic results on CAT(0) spaces). Throughout, tildes denote the obvious lifts to the universal cover.

Since $\tilde{S}$ is CAT(0), any $\tilde{p}, \tilde{q}$ are connected by a unique geodesic segment. Throughout, we will denote this segment by $[\tilde{p}, \tilde{q}]$.

Let GS be the set of all (parametrized) geodesics in S. That is,

$$GS = \{\gamma : \mathbb{R} \to S \mid \gamma \text{ is a local isometry}\}. \tag{1}$$

We endow GS with the following metric:

$$d_{GS}(\gamma_1, \gamma_2) = \inf_{\tilde{\gamma}_1, \tilde{\gamma}_2} \int_{-\infty}^{\infty} d_{\tilde{S}}(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t)) e^{-2|t|} \, dt, \tag{2}$$

where the infimum is taken over all lifts $\tilde{\gamma}_i$ of γ_i to $\tilde{GS}$ for $i = 1, 2$. GS serves as an analogue of the unit tangent bundle in our setting. (Indeed, for a Riemannian surface, GS is homeomorphic to $T^1 S$.) It is necessary to examine this more complicated space as geodesics in S are not determined by a tangent vector—they may branch apart from each other at points in Con. In this setting, the metric d_{GS} records the idea that two geodesics in GS are close if their images in S are nearby for all t in some large interval $[-T, T]$.

Geodesic flow on GS comes from shifting the parametrization of a geodesic:

$$(g_t \gamma)(s) = \gamma(s + t).$$

The normalizing factor 2 in our definition of d_{GS} ensures that g_t is a unit-speed flow with respect to d_{GS}. (Showing this is a completely straightforward computation, using the fact that $d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\gamma}(s + t)) = s$).

We recall two definitions of the K-property of an invariant measure. See Section 10.8 in [15] for a proof of the equivalence of these definitions (known as completely positive entropy and K-mixing, respectively) with the original definition of the K-property, as well as more details about other equivalent definitions.

Definition 2.1. A flow-invariant measure μ has the K-property if $(X, (g_t), \mu)$ has no nontrivial zero entropy factors (i.e., the Pinsker factor is trivial).

This definition can be reformulated as a statement about mixing in the following manner.

Definition 2.2. A flow-invariant measure μ has the K-property if for all $t \neq 0$, for all $k \geq 1$, and all measurable sets $A_0, A_1, \ldots, A_k$, we have

$$\lim_{n \to \infty} \sup_{B \in \mathcal{C}_n(A_1, \ldots, A_k)} \left| \mu(A_0 \cap B) - \mu(A_0)\mu(B) \right| = 0,$$

where $\mathcal{C}_n(A_1, \ldots, A_k)$ is the minimal σ-algebra generated by $g_{tr}(A_j)$ for $1 \leq j \leq k$ and natural $r \geq n$.

Remark. The K-property implies strong mixing of all orders. We recall that an invariant measure μ is strongly mixing of all orders if for all $k \geq 1$ and all measurable sets $A_0, A_1, \ldots, A_k$, we have

$$\lim_{t_1 \to \infty, \, t_{j+1} - t_j \to \infty} \mu(A_0 \cap g_{t_1}(A_1) \cap \ldots \cap g_{t_k}(A_k)) = \prod_{j=0}^{k} \mu(A_j).$$

A key tool in our analysis of the geodesic flow on S will be the turning angle of a geodesic at a cone point. We note that although S is not smooth at $p \in Con$, there is a well-defined space of directions at p, $S_p S$ and a well-defined notion of angle (see, e.g., [3, Ch. II.3]). In the angular metric, $S_p S$ is a circle of total circumference $\mathcal{L}(p)$.

Definition 2.3. Let $\gamma \in GS$. The turning angle of γ at time t is $\theta(\gamma, t) \in (-\frac{1}{2}\mathcal{L}(\gamma(t)), \frac{1}{2}\mathcal{L}(\gamma(t))]$ and is the signed angle between the segments $[\gamma(t - \delta), \gamma(t)]$ and $[\gamma(t), \gamma(t + \delta)]$ (for sufficiently small $\delta > 0$). A positive (resp. negative) sign for θ corresponds to a counterclockwise (resp. clockwise) rotation with respect to the orientation of $[\gamma(t - \delta), \gamma(t)]$.

Since γ is a geodesic, $|\theta(\gamma, t)| - \pi \geq 0$ for any $t \in \mathbb{R}$. If $\gamma(t) \notin Con$, then $\theta(\gamma, t) = \pi$.

Definition 2.4. We define the singular geodesics in S as

$$\text{Sing} = \{\gamma \in GS : |\theta(\tilde{\gamma}, t)| = \pi \quad \forall t \in \mathbb{R}\}.$$

Since Sing is defined in terms of properties of full geodesics, it is g_t-invariant. Geodesics not in Sing turn by some angle $\neq \pi$ at a cone point. This is an open condition, so Sing is closed and hence compact.

The geodesics in Sing either never encounter any cone points or, when they do, turn by angle exactly $\pm\pi$. They serve as an analogue of the singular set in the Riemannian setting of [4], that is, geodesics that remain entirely in zero-curvature regions of the surface. In both cases, the idea is that a singular geodesic never takes advantage of the geometric features of the surface (either its negative curvature regions or its large-angle cone points) to produce hyperbolic dynamical behavior. We note here a potentially confusing aspect of this terminology: a singular geodesic in this paper avoids the "singular", that is non-smooth, points of *Con*, or treats them as if they are not "singular".

We introduce some classical notions of thermodynamical formalism.

Definition 2.5.   Consider a function $\phi\colon GS \to \mathbb{R}$ that we refer as a *potential function*. The *pressure* for ϕ is

$$P(\phi) = \sup_{\mu} \left(h_{\mu}(g_t) + \int_{GS} \phi \, \mathrm{d}\mu \right),$$

where μ varies over all invariant Borel probability measures for g_t and $h_{\mu}(g_t)$ is the measure-theoretic entropy with respect to the geodesic flow.

An invariant Borel probability measure μ_{ϕ} (if it exists) such that

$$P(\phi) = h_{\mu_{\phi}}(g_t) + \int_{GS} \phi \, \mathrm{d}\mu_{\phi}$$

is an *equilibrium state* for ϕ.

Definition 2.6. $P(\mathrm{Sing}, \phi)$ is the pressure of the potential $\phi|_{\mathrm{Sing}}$ on the compact and flow-invariant set Sing (see Definition 2.4).

Below, we discuss some of the necessary definitions to apply the Climenhaga–Thompson machinery.

Definition 2.7.   Let $\varepsilon > 0$. The non-expansive set at scale ε for the flow g_t is

$$\mathrm{NE}(\varepsilon) = \{\gamma \in GS \mid \Gamma_{\varepsilon}(\gamma) \not\subset g_{[-s,s]}\gamma \text{ for all } s > 0\},$$

where

$$\Gamma_{\varepsilon}(\gamma) = \{\xi \in GS \mid d_{GS}(g_t\gamma, g_t\xi) \leq \varepsilon \quad \forall t \in \mathbb{R}\}.$$

The pressure of obstructions to expansivity for a potential ϕ is

$$P_{\exp}^{\perp}(\phi) = \lim_{\varepsilon \downarrow 0} \sup \left\{ h_{\mu}(g_1) + \int_{GS} \phi \, \mathrm{d}\mu \mid \mu(\mathrm{NE}(\varepsilon)) = 1 \right\},$$

where the supremum is taken over all g_t-invariant ergodic probability measures μ on GS such that $\mu(\mathrm{NE}(\varepsilon)) = 1$.

In other words, a geodesic is in the complement of $\mathrm{NE}(\varepsilon)$ if the only geodesics that stay ε close to it for all time are contained in its own orbit. A flow is expansive

if NE(ε) is empty for all sufficiently small ε. The presence of flat strips in our setting means our flow will not be expansive, but for small ε, the complement of NE(ε) will turn out to be a sufficiently rich set to use in our arguments.

In the interest of concision, we omit the formal definition of an orbit decomposition, referring instead to [10]. We will use a specific type of decomposition that has been studied in [5, 6], and we will primarily use results from those two papers. We note, however, that results from [10] hold for our decompositions as well, as it is written for a more general class of decomposition. We discuss this more in Section 8, where we will need to appeal to a few results directly from [10]. Identify a pair $(\gamma, t) \in GS \times [0, \infty)$ with the *orbit segment* $\{g_s\gamma \mid s \in [0, t]\}$. An orbit decomposition is a method of decomposing any orbit segment into three subsegments, a prefix, a central good segment, and a suffix. We denote the collections of these segments by $\mathcal{P}, \mathcal{G}$, and $\mathcal{S}$, respectively. The λ-decompositions that we use in this paper are orbit decompositions that decompose orbit segments based on a lower semicontinuous function λ. Our choices for the function λ and the associated parameter $\eta > 0$ will be discussed in detail in Section 3, but the idea is this. The function λ measures the amount of "hyperbolic" behavior seen by the geodesic; in accord with our intuition that cone points are the source of this behavior, λ will be based on turning angles at these points. A segment is "good" for our purposes (i.e., in $\mathcal{G}(\eta)$) if it experiences a lot of hyperbolicity; otherwise, it is in $\mathcal{P} = \mathcal{S}$:

- $\mathcal{G} = \mathcal{G}(\eta)$ consists of all (γ, t) such that the average value of λ over every initial and terminal segment of (γ, t) is at least η, and
- $\mathcal{P} = \mathcal{S} = \mathcal{B}(\eta)$ consists of all (γ, t) over which the average value of λ is less than η.

We can define both specification and the Bowen property for an arbitrary collection of orbit segments $\mathcal{G} \subset GS \times [0, \infty)$. In both cases, by taking $\mathcal{G} = GS \times [0, \infty)$, one retrieves the definitions for the full dynamical system.

Definition 2.8.   We say that $\mathcal{G}$ has *weak specification* if for all $\varepsilon > 0$, there exists $\tau > 0$ such that for any finite collection $\{(x_i, t_i)\}_{i=1}^n \subset \mathcal{G}$, there exists $y \in GS$ that ε-shadows the collection with transition times $\{\tau_i\}_{i=1}^n$ at most τ between orbit segments. In other words, for $1 \le i \le n$, there exists $\tau_i \in [0, \tau]$ and $y \in GS$ such that

$$d_{GS}(g_{t+s_i}y, g_t x_i) \le \varepsilon \text{ for } 0 \le t \le t_i,$$

where $s_k = \sum_{j=1}^{k-1} t_j + \tau_j$. We will refer to such τ as a *specification constant*.

We say that $\mathcal{G}$ has *strong specification* when we can always take each $\tau_j = \tau$ in the above definition.

Definition 2.9.   Given a potential $\phi : GS \to \mathbb{R}$, we say that ϕ has the Bowen property on $\mathcal{G}$ if there is some $\varepsilon > 0$ for which there exists a constant $K > 0$ such that

$$\sup \left\{ \left| \left| \int_0^t \phi(g_r x) - \phi(g_r y) \, dr \right| \right| \mid (x, t) \in \mathcal{G} \text{ and } d_{GS}(g_r y, g_r x) \le \varepsilon \text{ for } 0 \le r \le t \right\} \le K.$$

Remark.   If ϕ has the Bowen property on a collection of orbit segments $\mathcal{G}$ at some scale $\varepsilon > 0$, it in turn has the Bowen property on $\mathcal{G}$ at all smaller scales $\varepsilon' < \varepsilon$.

There is also a definition of topological pressure for collections of orbit segments. However, by using Theorem 1.1, we sidestep this complication.

Finally, we adapt a piece of terminology from flat surfaces to our somewhat more general setting.

Definition 2.10.   A geodesic segment with both endpoints in *Con* and no cone points in its interior is called a saddle connection. A saddle connection path is composed of saddle connections joined so that the turning angle at each cone point is at least π. Note that with this definition all saddle connection paths are geodesic segments.

2.2   Basic geometric results

In this section, we collect a few basic results on the geometry of S, $\tilde{S}$, GS, and $G\tilde{S}$ that will be used in our subsequent arguments.

The following two lemmas relate the metric d_{GS} to the metric d_S on the surface itself and will be useful for a number of our calculations below. First, we note that if two geodesics are close in GS, then they are close in S at time zero.

Lemma 2.11. ([13, Lemma 2.8])   For all $\gamma_1, \gamma_2 \in GS$,

$$d_S(\gamma_1(0), \gamma_2(0)) \le 2d_{GS}(\gamma_1, \gamma_2).$$

Furthermore, for $s, t \in \mathbb{R}$, $d_S(\gamma_1(s), \gamma_2(t)) \le 2d_{GS}(g_s\gamma_1, g_t\gamma_2)$.

Conversely, if two geodesics are close in S for a significant interval of time surrounding zero, then they are close in GS.

Lemma 2.12. ([13, Lemma 2.11])   Let ε be given and $a < b$ arbitrary. There exists $T = T(\varepsilon) > 0$ such that if $d_S(\gamma_1(t), \gamma_2(t)) < \varepsilon/2$ for all $t \in [a - T, b + T]$, then $d_{GS}(g_t\gamma_1, g_t\gamma_2) < \varepsilon$ for all $t \in [a, b]$. For small ε, we can take $T(\varepsilon) = -\log(\varepsilon)$.

A similar, and more specialized, result that we will need later in the paper (see the proof of Proposition 6.2) is the following.

Lemma 2.13.   Suppose that $d_S(\gamma_1(t), \gamma_2(t)) = 0$ for all $t \in [a, b]$. Then, for all $t \in [a, b]$, $d_{GS}(g_t\gamma_1, g_t\gamma_2) \leq e^{-2\min\{|t-a|, |t-b|\}}$.

Proof.   For any $x \geq 0$, $\int_x^\infty (s - x)e^{-2s}\, ds = \frac{1}{4}e^{-2x}$. In the setting of the lemma, since the distance between the geodesics is zero on $[a, b]$ and since geodesics move at unit speed,

$$d_{GS}(g_t\gamma_1, g_t\gamma_2) \leq \int_{-\infty}^a 2(a - s)e^{-2|t-s|}\, ds + \int_b^\infty 2(s - b)e^{-2|t-s|}\, ds.$$

Quick changes of variables show that this is equal to $\int_{|t-a|}^\infty 2(s - |t-a|)e^{-2s}\, ds + \int_{|t-b|}^\infty 2(s - |t-b|)e^{-2s}\, ds = \frac{1}{2}(e^{-2|t-a|} + e^{-2|t-b|})$, and the lemma follows.   ∎

The geodesic flow has the following Lipschitz property.

Lemma 2.14. ([14, Lemma 2.5])   Fix a $T > 0$. Then, for any $t \in [0, T]$, and any pair of geodesics $\gamma, \xi \in GS$,

$$d_{GS}(g_t\gamma, g_t\xi) < e^{2T} d_{GS}(\gamma, \xi).$$

We need the following four geometric facts.

Lemma 2.15.

(a)   There exists some $d_0 > 0$ such that $\tilde{S}$ contains no flat $d_0 \times d_0$ square.

(b)   There exists some $\eta_0 > 0$ such that the excess angle at every cone point in S is at least η_0.

(c)   There exists some $\ell_0 > 0$ such that the length of every saddle connection is at least ℓ_0.

(d)   There exists some $\theta_0 > 0$ such that the excess angle at every cone point in S is at most θ_0.

Proof.   These follow immediately from the compactness of S and the fact that S having genus at least two implies $Con \neq \emptyset$.   ∎

We note here that Sing is the source of the non-expansivity for our geodesic flow.

Lemma 2.16.   For all $\varepsilon > 0$ less than half the injectivity radius of S, $\mathrm{NE}(\varepsilon) \subset \mathrm{Sing}$.

Proof.   Suppose $\gamma \in \mathrm{NE}(\varepsilon)$ and that ε is smaller than half the injectivity radius of S. Then, there exists $\xi \in GS$ which is not in the orbit of γ such that $d_{GS}(g_t\gamma, g_t\xi) \leq \varepsilon$ for all $t \in \mathbb{R}$. By Lemma 2.11, $d_S(\gamma(t), \xi(t)) \leq 2\varepsilon$ for all $t \in \mathbb{R}$. In particular, using our assumption on ε, there exist lifts $\tilde{\gamma}$ and $\tilde{\xi}$ such that $d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\xi}(t)) \leq 2\varepsilon$ for all $t \in \mathbb{R}$. By the flat strip theorem [1, Corollary 5.8 (ii)], there is an isometric embedding $\mathbb{R} \times [a, b] \to \tilde{S}$ sending $\mathbb{R} \times \{a\}$ to the image of $\tilde{\gamma}$ and $\mathbb{R} \times \{b\}$ to the image of $\tilde{\xi}$.

Since $\tilde{\xi}$ is not in the orbit of $\tilde{\gamma}$, we must have $a \neq b$ and the isometrically embedded strip is nondegenerate. But this immediately implies that for all t, $|\theta(\tilde{\gamma}, t)| = \pi$ as $\tilde{\gamma}$ always turns at angle π on the side to which the embedded flat strip lies. Therefore, $\gamma \in \mathrm{Sing}$.   ∎

Recall that a flow is called entropy-expansive if for sufficiently small ε, $\sup\{h_{top}(g_t|_{\Gamma_\varepsilon(\gamma)}) \mid \gamma \in GS\} = 0$.

Lemma 2.17.   ([20, Lemma 20])   The geodesic flow in our setting is entropy-expansive.

Proof.   This is proven by Ricks [20] for geodesic flow on a CAT(0) space. This covers our setting, but Ricks uses a slightly different definition of the metric on GS than we do, so we outline the argument here.

Fix ε less than half the injectivity radius of S. Lift γ to $\tilde{\gamma} \in G\tilde{S}$. Any geodesics $\xi \in \Gamma_\varepsilon(\gamma)$ lift to $\tilde{\xi} \in \Gamma_\varepsilon(\tilde{\gamma})$. They are either of the form $g_t\tilde{\gamma}$ for $|t| < \varepsilon$ or are parallel to $\tilde{\gamma}$ in a flat strip containing $\tilde{\gamma}$. The flow on $\Gamma_\varepsilon(\tilde{\gamma})$ is thus isometric, and so $h_{top}(g_t|_{\Gamma_\varepsilon(\gamma)}) = 0$.   ∎

Lemma 2.18.   Given any closed geodesic $\gamma \subset S$, there is a closed saddle connection path that is homotopic to γ and has the same length as γ.

Proof.   Assume γ contains a point $p \in Con$. Then the desired closed saddle connection path is the geodesic that starts at p and traces γ.

Suppose $\gamma \subset S \setminus Con$, and so $\tilde{\gamma} \subset \tilde{S} \setminus \widetilde{Con}$. Fix an orientation of $\tilde{\gamma}$, and consider the variation $\tilde{\gamma}_r$ of curves given by sliding $\tilde{\gamma}$ to its left (so the variational field is

perpendicular to $\tilde{\gamma}$ and to its left with respect to $\tilde{\gamma}$'s orientation). Since $\tilde{\gamma} \subset \tilde{S} \setminus \widetilde{Con}$ and γ is closed, there is a nonzero lower bound on the distance from $\tilde{\gamma}$ to $\widetilde{Con}$. Therefore, for all sufficiently small r, $\tilde{\gamma}_r$ is defined. The projections to S, γ_r, and γ form the boundary of a flat cylinder in S. Thus, γ_r is a geodesic with length equal to that of γ.

Let r^* be the supremum of all $r > 0$ for which $\tilde{\gamma}_\rho$ is defined for all $\rho \in [0, r]$. Note that if no supremum exists, $\tilde{\gamma}$ bounds a flat half-space in $\tilde{S}$, which contains a fundamental domain for S since S is compact. This would imply S is flat (with *no* cone points), a contradiction. Therefore, letting $r \to r^*$ from below, $\tilde{\gamma}_r$ limits uniformly on a path, and therefore necessarily a geodesic, containing at least one point in $\widetilde{Con}$ with the same length as $\tilde{\gamma}$. The image of this curve in S (with appropriate parametrization) is the saddle connection path we want.   ∎

In the proof of Lemma 2.20 and in some later proofs, we will use the following construction.

Definition 2.19.   Let $\tilde{\gamma}$ be a geodesic segment in $\tilde{S}$ with endpoint p. The cone around $\tilde{\gamma}$ with vertex p and angle ψ is the set of all points q in $\tilde{S}$ such that the unique geodesic segment joining p and q makes angle $\leq \psi$ with $\tilde{\gamma}$ at p. (In Section 3, Figure 2 shows such cones in the context of the proof of Lemma 3.8.)

Lemma 2.20.   For any $\zeta \in Con$, there exists a closed geodesic α passing through ζ with turning angle greater than π at ζ.

Proof.   Let ζ be a cone point with $\mathcal{L}(\zeta) = 2\pi + \beta$ for $\beta > 0$. Lift ζ to $\tilde{\zeta}$ in $\tilde{S}$, and let $\tilde{c}$ be a geodesic with $\tilde{c}(0) = \tilde{\zeta}$ and turning angle $\theta(\tilde{c}, 0) = \pi + \frac{\beta}{2}$. Let C_1 be the cone around $\tilde{c}(-\infty, 0)$ with vertex $\tilde{\zeta} = \tilde{c}(0)$ and angle $\frac{\beta}{8}$; let C_2 be the cone around $\tilde{c}(0, \infty)$ with vertex $\tilde{\zeta} = \tilde{c}(0)$ and angle $\frac{\beta}{8}$. By construction, any geodesic connecting a point in $C_1 \setminus \{\tilde{\zeta}\}$ to a point in $C_2 \setminus \{\tilde{\zeta}\}$ must pass through $\tilde{\zeta}$ with turning angle $\geq \pi + \frac{\beta}{4}$.

Let $\mathcal{F}$ be a fundamental domain contained in C_1. Let $g \in \pi_1(S)$ be such that $g\mathcal{F} \subset C_2$. ($\mathcal{F}$ and g exist as both C_1 and C_2 contain arbitrarily large balls and S is compact.) Let α be the closed geodesic representative of g in S. (It will become clear in a moment why α is unique up to parametrization.) Lift α to $\tilde{\alpha}$ with $\tilde{\alpha}(0) \in \mathcal{F}$. Then $\tilde{\alpha}(\ell(\alpha)) \in g\mathcal{F}$. As noted above, this forces $\tilde{\alpha}$ to pass through $\tilde{\zeta}$ and turn with angle $> \pi$. Therefore, α is the desired geodesic (and it is unique up to parametrization since it cannot belong to a flat cylinder).   ∎
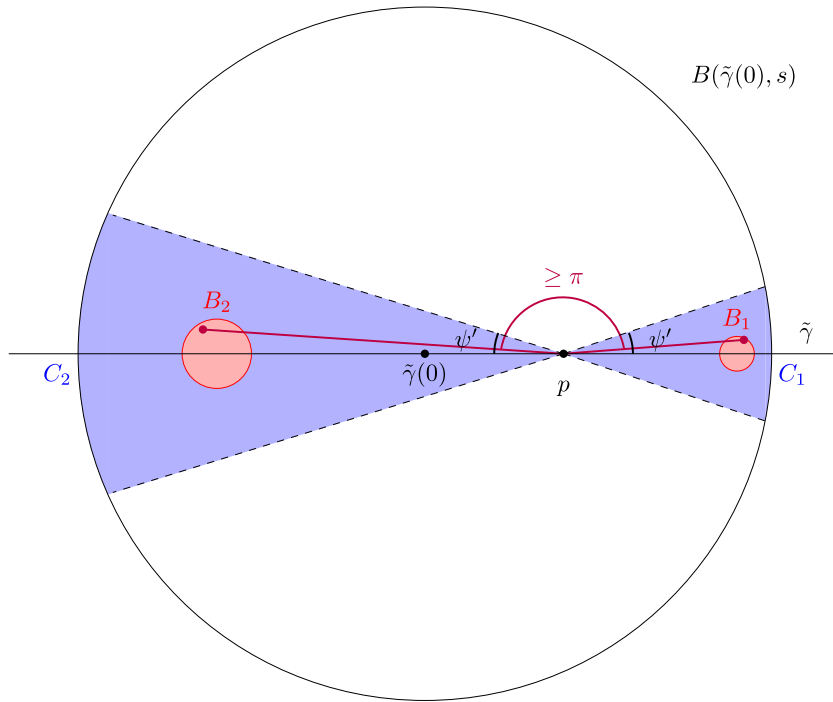
Fig. 2. The argument for Case 1 in Lemma 3.8. The geodesic segments connecting points in B_2 and B_1 meet at the cone point p with angle $\geq \pi$ on both sides. Any geodesic connecting points in B_2 and B_1 must run through p.

3 The λ-Decomposition

We now turn to the main arguments of the paper. First, following the ideas in [4], we establish the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ as a "λ-decomposition" using the function λ in Definition 3.3 that is defined through two auxiliary functions that view the stable and unstable parts of any given geodesic. Throughout this section, fix $s > 0$ such that $2s$ is less than the shortest saddle connection of S. Below, we omit in the notation the dependence of functions on s.

Definition 3.1.   We define $\lambda^{uu} \colon GS \to [0, \infty)$ by

$$\lambda^{uu}(\gamma) = \frac{|\theta(\gamma, c)| - \pi}{\max\{s, c\}},$$

where $c \geq 0$ is the 1st time that $\gamma(c)$ hits a cone point and turns with angle strictly greater than π (naturally, we set $\lambda^{uu}(\gamma) = 0$ in case $c = \infty$).

Definition 3.2.   We define $\lambda^{ss} : GS \to [0, \infty)$ by

$$\lambda^{ss}(\gamma) = \frac{|\theta(\gamma, c)| - \pi}{\max\{s, |c|\}},$$

where $c \leq 0$ is the most recent time that $\gamma(c)$ has hit a cone point and turned with angle strictly greater than π (naturally, we set $\lambda^{ss}(\gamma) = 0$ in case $c = -\infty$).

We now define our function λ so that near cone points at which geodesics turn with angle greater than π, it measures the turning angle at that cone point (multiplied by a constant), and far from a cone point, it measures both distance and turning angle from both the previous and next cone point.

Definition 3.3.   Let λ^{uu} and λ^{ss} be functions defined in Definitions 3.1 and 3.2, respectively. We define $\lambda : GS \to [0, \infty)$ by

$$\lambda(\gamma) = \begin{cases} \lambda^{ss}(\gamma) & \text{if there exists } c \in (-s, 0] \text{ such that } |\theta(\gamma, c)| - \pi > 0, \\ \lambda^{uu}(\gamma) & \text{if there exists } c \in [0, s) \text{ such that } |\theta(\gamma, c)| - \pi > 0, \\ \min\{\lambda^{ss}(\gamma), \lambda^{uu}(\gamma)\} & \text{otherwise.} \end{cases}$$

Observe that it is well defined when $\gamma(0)$ is a cone point, as in that case, $\lambda^{uu}(\gamma) = \lambda^{ss}(\gamma)$.

We prove several properties of λ.

Proposition 3.4.   If $\lambda(\gamma) = 0$, then $\lambda(g_t\gamma) = 0$ either for all $t \geq 0$ or for all $t \leq 0$.

Proof.   If $\lambda(\gamma) = 0$, then γ does not turn at a cone point in the interval $(-s, s)$, and so, $\lambda^{uu}(\gamma) = 0$ or $\lambda^{ss}(\gamma) = 0$. In the 1st case, this implies that γ never turns at a cone point in the future. Therefore, for all $t \geq 0$, $\lambda(g_t\gamma) = \lambda^{uu}(g_t\gamma) = 0$. A similar argument holds with $t \leq 0$ if $\lambda^{ss}(\gamma) = 0$. ∎

As corollaries, we have the following.

Corollary 3.5. $\bigcap_{t \in \mathbb{R}} g_t \lambda^{-1}(0) = \text{Sing}$.

Corollary 3.6.   If $\lambda(\gamma) = 0$, then $d(g_t\gamma, \text{Sing}) \to 0$ either as $t \to \infty$ or as $t \to -\infty$.

Proof.   Without loss of generality, assume $\lambda(g_t\gamma) = 0$ for all $t \geq 0$. Then, γ does not turn at a cone point in $[0, \infty)$, and we can define $r := \max\{t : |\theta(\gamma, t)| > \pi\}$ to be the most recent cone point in the past at which γ turns. Define a singular geodesic γ_{Sing} as $\gamma_{\text{Sing}}(t) = \gamma(t)$ for all $t > r$, and for all cone points $t \leq r$, γ_{Sing} turns with angle π. Then, $g_t\gamma$ and $g_t\gamma_{\text{Sing}}$ agree on increasingly long intervals, and by Lemma 2.13 for $t > r$,

$$d_{GS}(g_t\gamma, \text{Sing}) \leq d_{GS}(g_t\gamma, g_t\gamma_{\text{Sing}}) \leq e^{-2(t-r)}.$$

The proof if $\lambda(g_t\gamma) = 0$ holds similarly, but sending $t \to -\infty$ instead.   ∎

Furthermore, this allows us to show that the pressure gap for the product flow (condition (3) of Theorem 1.1) is implied by the pressure gap $P(\text{Sing}, \phi) < P(\phi)$ that we will establish in Section 7.

Proposition 3.7 (Following [5, Proposition 5.1]).   Setting $\Phi(x, y) = \phi(x) + \phi(y)$ and $\tilde{\lambda}(x, y) = \lambda(x)\lambda(y)$, we have $P(\bigcap_{t\in\mathbb{R}}(g_t \times g_t)(\tilde{\lambda})^{-1}(0), \Phi) \leq P(\phi) + P(\text{Sing}, \phi)$. In particular, if $P(\text{Sing}, \phi) < P(\phi)$, then $P(\bigcap_{t\in\mathbb{R}}(g_t \times g_t)(\tilde{\lambda})^{-1}(0), \Phi) < 2P(\phi)$.

Proof.   The variational principle [21, Theorem 9.10] tells us that

$$P\left(\bigcap_{t\in\mathbb{R}}(g_t \times g_t)(\tilde{\lambda}^{-1}), \Phi\right) = \sup\left\{P_\nu(\Phi) \,\Big|\, \nu \text{ is flow invariant and } \nu\left(\bigcap_{t\in\mathbb{R}}(g_t \times g_t)(\tilde{\lambda}^{-1})\right) = 1\right\},$$

where $P_\nu(\Phi) := h_\nu(g_1 \times g_1) + \int \Phi \, d\nu$ denotes the measure-theoretic pressure of $(\bigcap_{t\in\mathbb{R}}(g_t \times g_t)(\tilde{\lambda}^{-1}), (g_t \times g_t), \Phi, \nu)$. More generally, this relationship holds for any continuous flow, continuous potential, and compact, flow-invariant subset.

Consequently, we let ν be an invariant measure supported on $\bigcap_{t\in\mathbb{R}}(g_t \times g_t)(\tilde{\lambda})^{-1}(0)$ and let

$$A = \bigcap_{t\in\mathbb{R}}(g_t \times g_t)(\tilde{\lambda})^{-1}(0) \cap (\text{Reg} \times \text{Reg}).$$

We will show that $\nu(A) = 0$ by showing that it contains no recurrent points. Assume for contradiction that $(\gamma_1, \gamma_2) \in A$ is a recurrent point, and then assume without loss of generality that $\lambda(\gamma_1) = 0$. Since $\gamma_1 \notin \text{Sing}$, it follows that $d_{GS}(\gamma_1, \text{Sing}) = c > 0$, which from recurrence, implies that there exists a sequence $t_k \to \infty$ such that $d_{GS}(g_{t_k}\gamma_1, \text{Sing}) > \frac{c}{2}$, with a similar claim holding in backwards time. However, we also know that $d_{GS}(g_t\gamma_1, \text{Sing}) \to 0$ as $t \to \infty$, or as $t \to -\infty$ by Corollary 3.6. Thus, we have

arrived at a contradiction. Hence, ν is supported on the complement of Reg $\times$ Reg, which is (Sing $\times$ GS) $\cup$ (GS $\times$ Sing).

Thus,

$$P\left(\bigcap_{t\in\mathbb{R}}(g_t \times g_t)(\tilde{\lambda}^{-1}(0)), \Phi\right) \le P\left((\text{Sing} \times GS) \cup (GS \times \text{Sing}), \Phi\right) \le P(\text{Sing}, \phi) + P(GS, \phi).$$

The 1st inequality is by the variational principle. The 2nd inequality is due to the fact that the pressure of the union of two compact invariant sets is the maximum of the pressure of each individual set [18, Theorem 11.2(3)], and in this case, the pressure of each component of the union is at most $P(\text{Sing}, \phi) + P(GS, \phi)$ by [21, Theorem 9.8(v)].   ■

We have also constructed λ so that it is lower semicontinuous.

Lemma 3.8.   Let $s > 0$ be such that $2s$ is less than the shortest saddle connection of S. Then, λ defined in Definition 3.3 is lower semicontinuous.

Proof.   Let $\gamma \in GS$. We show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\lambda(\gamma) - \varepsilon < \lambda(\xi)$ for all $\xi \in GS$ such that $d_{GS}(\gamma, \xi) < \delta$. To ease the arguments below slightly, we work in $\tilde{S}$ with lifts $\tilde{\gamma}, \tilde{\xi}$ so that $d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\xi}) = d_{GS}(\gamma, \xi)$. Recall that by Lemma 2.11, if $d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\xi}) < \delta$ then $d_{\tilde{S}}(\tilde{\gamma}(0), \tilde{\xi}(0)) < 2\delta$.

If $\lambda(\gamma) = 0$, then we are done as λ is a non-negative function. Therefore, for the rest of the argument, we assume that $\lambda(\gamma) > 0$.

Case 1: Suppose there exists $c \in (-s, s)$ such that $\psi := |\theta(\tilde{\gamma}, c)| - \pi > 0$. Denote $\tilde{\gamma}(c) = p$. We show that there exists $\delta > 0$ such that $p \in \tilde{\xi}((-s, s))$.

Let C_1 be the cone around $\tilde{\gamma}((c, s))$ with vertex p and angle $\psi' = \min\{\frac{\psi}{2}, \frac{\pi}{4}\}$. Let C_2 be the cone around $\tilde{\gamma}((-s, c))$ with vertex p and angle ψ'. (See Figure 2.) Set

$$\delta = \frac{1}{2} \min\left\{ \frac{1}{8}e^{-2s}(s - |c|)\sin\psi', \quad \frac{1}{2}(s - d_S(\tilde{\gamma}(0), p)), \quad \min\{1, \varepsilon s/8\}(2 + e^{2|c|})^{-1} \int_0^{s-|c|} 2te^{-2t}\, dt \right\}. \tag{3}$$

Then, we choose $u_1 = \frac{c+s}{2}, u_2 = \frac{c-s}{2}$ and $\delta_1 = \frac{s-c}{4}\sin\psi', \delta_2 = \frac{c+s}{4}\sin\psi' > 0$ so that $B_1 := B(\tilde{\gamma}(u_1), \delta_1) \subset C_1 \cap B(\tilde{\gamma}(0), s)$ and $B_2 := B(\tilde{\gamma}(u_2), \delta_2) \subset C_2 \cap B(\tilde{\gamma}(0), s)$. By Lemmas 2.11 and 2.14, since $\delta < \frac{1}{8}e^{-2s}(s - |c|)\sin\psi'$, if $d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\xi}) < \delta$, then $\tilde{\xi}$ passes through B_1 and B_2. Since any two points in a CAT(0)-space are connected by a unique geodesic segment and by our construction of C_1 and C_2, we obtain that if $d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\xi}) < \delta$, then $\tilde{\xi}$ passes through p. Furthermore, since $2\delta + d_S(\tilde{\gamma}(0), p) < s$, by Lemma 2.11 and the triangle inequality

for the triangle with vertices $\tilde{\xi}(0)$, $\tilde{\gamma}(0)$, and p, we have $p \in \tilde{\xi}((-s, s))$ if $d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\xi}) < \delta$. Let $t_0 \in (-s, s)$ be such that $\tilde{\xi}(t_0) = p$. Moreover, $|t_0 - c| \leq 2\delta$.

By the triangle inequality,

$$d_{\tilde{S}}(\tilde{\xi}(t_0 + t), \tilde{\gamma}(c + t)) \leq d_{\tilde{S}}(\tilde{\xi}(t_0 + t), \tilde{\xi}(c + t)) + d_{\tilde{S}}(\tilde{\xi}(c + t), \tilde{\gamma}(c + t)) = |t_0 - c|$$
$$+ d_{\tilde{S}}(\tilde{\xi}(c + t), \tilde{\gamma}(c + t)).$$

Let $\tilde{\xi}_1 = g_{t_0}\tilde{\xi}$ and $\tilde{\gamma}_1 = g_c\tilde{\gamma}$. Then, by the above inequality and Lemma 2.14,

$$d_{G\tilde{S}}(\tilde{\xi}_1, \tilde{\gamma}_1) \leq 2\delta + e^{2|c|}\delta = (2 + e^{2|c|})\delta. \tag{4}$$

Moreover, for all $t \in (0, s - c]$, we obtain that

$$d_{\tilde{S}}(\tilde{\xi}_1(t), \tilde{\gamma}_1(t)) = \begin{cases} 2t & \text{if} \quad \alpha \geq \pi, \\ 2t\sin(\alpha/2) & \text{if} \quad 0 \leq \alpha \leq \pi, \end{cases}$$

where α is the (unsigned) angle between the outward trajectories of $\tilde{\gamma}_1$ and $\tilde{\xi}_1$ from the cone point p.

If $\alpha \geq \pi$, then $d_{G\tilde{S}}(\tilde{\xi}_1, \tilde{\gamma}_1) \geq \int_0^{s-c} 2te^{-2t}dt$, which is not possible by (4) and the choice of δ (see (3)).

Consider $\alpha \in [0, \pi)$. Then we have that

$$\sin(\alpha/2) < \delta(2 + e^{2|c|})\left(\int_0^{s-c} 2te^{-2t}\,dt\right)^{-1}. \tag{5}$$

Let β be the (unsigned) angle between the inward trajectories $\tilde{\gamma}_1$ and $\tilde{\xi}_1$ at p. Similarly to the argument above, we obtain that for δ as defined in (3),

$$\sin(\beta/2) < \delta(2 + e^{2|c|})\left(-\int_{-s-c}^0 2te^{2t}\,dt\right)^{-1}. \tag{6}$$

Using (5) and (6),

$$|\lambda(\gamma) - \lambda(\xi)| = \frac{1}{s}\left||\theta(\tilde{\gamma}, c)| - |\theta(\tilde{\xi}, t_0)|\right| \leq \frac{1}{s}(\alpha + \beta) \leq C\delta,$$

where $C = \frac{8}{s}(2 + e^{2|c|})\left(\int_0^{s-|c|} 2te^{-2t}\,dt\right)^{-1}$. Thus, for our choice of δ (see (3)), we have $|\lambda(\gamma) - \lambda(\xi)| < \varepsilon$.

Case 2: Assume there exists $c_1 \leq -s$ and $c_2 \geq s$ such that $\psi_1 := |\theta(\tilde{\gamma}, c_1)| - \pi > 0$ and $\psi_2 := |\theta(\tilde{\gamma}, c_2)| - \pi > 0$. Denote $\tilde{\gamma}(c_1) = p_1$ and $\tilde{\gamma}(c_2) = p_2$.

Let C_1 be the cone around the segment $\tilde{\gamma}((c_1, -s])$ if $c_1 \neq -s$ or $\tilde{\gamma}((-2s, -s))$ otherwise with vertex p_1 and angle $\psi' = \frac{\min\{\psi_1, \psi_2, \pi\}}{4}$. Let C_2 be the cone around the segment $\tilde{\gamma}([s, c_2])$ if $c_2 \neq s$ or $\tilde{\gamma}((s, 2s))$ otherwise, with vertex p_2 and angle ψ'. Set

$$c = \min\{|c_1|, c_2\} \text{ and } \delta = \frac{1}{2} \min\left\{\frac{1}{8} e^{-2s}(c - s) \sin \psi', \quad \min\{1, \varepsilon c/8\}(2e^{2c} + 1)^{-1} \int_c^\infty 2t e^{-2t}\, dt\right\}. \tag{7}$$

Similar to Case 1, by Lemmas 2.11 and 2.14 and the choice of δ in (7), if $d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\xi}) < \delta$ then $\tilde{\xi}$ passes through p_1 and p_2. In particular, $\tilde{\gamma}$ and $\tilde{\xi}$ share a geodesic connecting p_1 and p_2. Therefore, there exists d such that $g_d \tilde{\xi}(t) = \tilde{\gamma}(t)$ for $t \in [c_1, c_2]$. Let t_1 and t_2 be such that $\tilde{\xi}(t_1) = p_1$ and $\tilde{\xi}(t_2) = p_2$. Then, $|t_1 - c_1| \leq 2e^{2|c_1|}\delta$ and $|t_2 - c_2| \leq 2e^{2c_2}\delta$ so $|d| \leq 2e^{2c}\delta$. Moreover, by the triangle inequality,

$$d_{G\tilde{S}}(g_d \tilde{\xi}, \tilde{\gamma}) \leq (2e^{2c} + 1)\delta. \tag{8}$$

Let α_1 and α_2 be the (unsigned) angles between the inward and outward trajectories of $g_d \tilde{\xi}$ and $\tilde{\gamma}$ at p_1 and p_2, respectively. Similarly to Case 1, for our choice of δ, we have $0 \leq \alpha_1, \alpha_2 \leq \pi$,

$$\sin(\alpha_1/2) \leq \delta(2e^{2c} + 1)\left(-\int_{c_2}^\infty 2t e^{2t}\, dt\right)^{-1}$$

and

$$\sin(\alpha_2/2) \leq \delta(2e^{2c} + 1)\left(\int_{c_1}^\infty 2t e^{-2t}\, dt\right)^{-1}.$$

Therefore,

$$|\lambda^{ss}(\gamma) - \lambda^{ss}(\xi)| \leq C\delta \quad \text{and} \quad |\lambda^{uu}(\gamma) - \lambda^{uu}(\xi)| \leq C\delta,$$

where $C = \frac{8}{c}(2e^{2c} + 1)\left(\int_c^\infty 2t e^{-2t} dt\right)^{-1}$.

Thus, if $t_1 = c_1 + d \leq -s$ and $t_2 = c_2 + d \geq s$, then $\lambda(\xi) = \min\{\lambda^{ss}(\xi), \lambda^{uu}(\xi)\}$ and we have $|\lambda(\gamma) - \lambda(\xi)| \leq C\delta < \varepsilon$.

Otherwise, $\lambda(\xi) \geq \min\{\lambda^{ss}(\xi), \lambda^{uu}(\xi)\}$ and we have $\lambda(\xi) \geq \lambda(\gamma) - C\delta > \lambda(\gamma) - \varepsilon$. ∎

Remark.   Note that for this construction of λ, we do not in general have upper semicontinuity. To see this, consider a geodesic γ that turns with angle greater than π at times $-s$ and c for some $c > 0$. Then, for all $r \in (0, s]$, $\lambda(g_{-r}\gamma) = \lambda^{ss}(g_{-r}\gamma)$, while

$\lambda(\gamma) = \min\{\lambda^{ss}(\gamma), \lambda^{uu}(\gamma)\}$. Therefore, if $\lambda^{uu}(\gamma) < \lambda^{ss}(\gamma)$, we have that

$$\lambda(\gamma) = \lambda^{uu}(\gamma) < \lambda^{ss}(\gamma) = \lim_{r \downarrow 0} \lambda^{ss}(g_{-r}\gamma) = \lim_{r \downarrow 0} \lambda(g_{-r}\gamma).$$

This contradicts upper semicontinuity of λ.

Following Section of [4], or Definition 3.4 in [5] (and formalizing the idea presented in Section 2.1), we define

$$\mathcal{G}(\eta) = \left\{ (\gamma, t) \mid \int_0^\rho \lambda(g_u(\gamma)) \, du \geq \eta\rho \quad \text{and} \quad \int_0^\rho \lambda(g_{-u}g_t(\gamma)) \, du \geq \eta\rho \quad \text{for} \quad \rho \in [0, t] \right\}$$

and

$$\mathcal{B}(\eta) = \left\{ (\gamma, t) \mid \int_0^\rho \lambda(g_u(\gamma)) \, du < \eta\rho \right\}.$$

The decomposition we will take is $(\mathcal{P}, \mathcal{G}, \mathcal{S}) = (\mathcal{B}(\eta), \mathcal{G}(\eta), \mathcal{B}(\eta))$ for a sufficiently small value of η that will be determined below. We reiterate that because of our choice of decomposition, we do not need to consider the sets of orbit segments denoted by $[\mathcal{P}], [\mathcal{S}]$ because of [5, Lemma 3.5].

While near cone points, positivity of λ only gives us information about the closest cone point, and far from cone points, it gives us information about cone points on both sides. The following propositions help us quantify these relationships. Let θ_0 be as in Lemma 2.15(d).

Proposition 3.9. If $\lambda(\gamma) > \eta$, then there is a cone point in $\gamma[-\frac{\theta_0}{2\eta}, \frac{\theta_0}{2\eta}]$ with turning angle at least $s\eta$ away from $\pm\pi$. In particular, if $(\gamma, t) \in \mathcal{G}(\eta)$, then there exist $t_1, t_2 \in [-\frac{\theta_0}{2\eta}, \frac{\theta_0}{2\eta}]$ such that $\gamma(t_1), \gamma(t + t_2) \in \text{Con}$, with the turning angles at these cone points at least $s\eta$ away from $\pm\pi$.

Proof. Since $\lambda(\gamma) > \eta$, either $\lambda^{uu}(\gamma) > \eta$ or $\lambda^{ss}(\gamma) > \eta$. If $\lambda^{uu}(\gamma) > \eta$, then by Definition 3.3, there is a $c \geq 0$ such that $\gamma(c) \in Con$ and $\lambda^{uu}(\gamma) = \frac{|\theta(\gamma,c)| - \pi}{\max\{s,c\}}$. The turning angle at $\gamma(c)$ satisfies $|\theta(\gamma, c)| - \pi \leq \theta_0/2$. Thus,

$$\eta < \lambda^{uu}(\gamma) \leq \frac{|\theta(\gamma, c)| - \pi}{c} \leq \frac{\theta_0/2}{c}$$

and $0 \leq c \leq \frac{\theta_0}{2\eta}$. Furthermore,

$$\eta < \lambda^{uu}(\gamma) \leq \frac{|\theta(\gamma, c)| - \pi}{s}$$

so the turning angle of γ at c differs from π by at least $s\eta$.

A similar argument applies if $\lambda^{ss}(\gamma) > \eta$. ∎

Finally, we collect a statement we will need in Section 7.

Lemma 3.10. Given any $\eta > 0$, there exists a $\delta > 0$ such that $\lambda(\gamma) < \eta$ for all $\gamma \in B(\text{Sing}, 2\delta)$.

Proof. Let $\eta > 0$ be given, and suppose without generality, it is small enough that $\frac{s\eta}{32} < 1$. We argue in $\tilde{S}$. Suppose $\tilde{\gamma} \in B(\text{Sing}, 2\delta)$ and, in particular, that $\tilde{\xi} \in \text{Sing}$ with $d_{GS}(\tilde{\gamma}, \tilde{\xi}) < 2\delta$. We choose $\delta < \frac{s\theta_0}{64e^{4\theta_0/\eta}}$ and toward a contradiction suppose that $\lambda(\tilde{\gamma}) > \frac{\eta}{2}$. (Recall that s is specified in Lemma 3.8, and η_0 is specified in Lemma 2.15(d).)

Since $\lambda(\tilde{\gamma}) > \frac{\eta}{2}$, by Proposition 3.9, there exists a cone point in $\tilde{\gamma}[-\frac{\theta_0}{\eta}, \frac{\theta_0}{\eta}]$ at which $\tilde{\gamma}$ turns with angle at least $\frac{s\eta}{2}$ away from $\pm\pi$. Say $\tilde{\gamma}$ hits that cone point at time $t_0 \in [-\frac{\theta_0}{\eta}, \frac{\theta_0}{\eta}]$.

As $d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\xi}) < 2\delta$, by Lemma 2.14,

$$d_{G\tilde{S}}\left(g_{-\frac{2\theta_0}{\eta}}\tilde{\gamma}, g_{-\frac{2\theta_0}{\eta}}\tilde{\xi}\right) < 2\delta e^{\frac{4\theta_0}{\eta}} \quad \text{and} \quad d_{G\tilde{S}}\left(g_{\frac{2\theta_0}{\eta}}\tilde{\gamma}, g_{\frac{2\theta_0}{\eta}}\tilde{\xi}\right) < 2\delta e^{\frac{4\theta_0}{\eta}}.$$

Then, by Lemma 2.11,

$$d_{\tilde{S}}\left(\tilde{\gamma}\left(-\frac{2\theta_0}{\eta}\right), \tilde{\xi}\left(-\frac{2\theta_0}{\eta}\right)\right) < 4\delta e^{\frac{4\theta_0}{\eta}} \quad \text{and} \quad d_{\tilde{S}}\left(\tilde{\gamma}\left(\frac{2\theta_0}{\eta}\right), \tilde{\xi}\left(\frac{2\theta_0}{\eta}\right)\right) < 4\delta e^{\frac{4\theta_0}{\eta}}.$$

Consider the geodesic segment c connecting $\tilde{\xi}(-\frac{2\theta_0}{\eta})$ and $\tilde{\gamma}(t_0)$. The segment c and $\tilde{\gamma}[-\frac{2\theta_0}{\eta}, t_0]$ agree at t_0 and at time $-\frac{2\theta_0}{\eta}$, at least $\frac{\theta_0}{\eta}$ away with respect to $d_{\tilde{S}}$, are at most $4\delta e^{\frac{4\theta_0}{\eta}}$ apart. Comparing to a Euclidean triangle and using the CAT(0) property, the angle between these segments at $\tilde{\gamma}(t_0)$ is at most $2\sin^{-1}[(4\delta e^{\frac{4\theta_0}{\eta}})/(2\frac{\theta_0}{\eta})]$. By our choice of δ, this is less than $2\sin^{-1}[\frac{s\eta}{32}] < \frac{s\eta}{8}$. The same argument applies to the angle between $\tilde{\gamma}[t_0, \frac{2\theta_0}{\eta}]$ and the segment c' from $\tilde{\gamma}(t_0)$ to $\tilde{\xi}(\frac{2\theta_0}{\eta})$.

At t_0, $\tilde{\gamma}$ turns with angle at least $\frac{s\eta}{2}$ away from $\pm\pi$. Therefore, the concatenation of c with c' turns with angle at least $\pi + \frac{s\eta}{4}$ on both sides and hence is geodesic. By

uniqueness of geodesic segments in $\tilde{S}$, $\xi[-\frac{2\theta_0}{\eta}, \frac{2\theta_0}{\eta}]$ must agree with this concatenation. But this contradicts the fact that $\xi \in \text{Sing}$. Therefore, $\lambda(\gamma) \leq \frac{\eta}{2} < \eta$ as desired.   ∎

4 $\mathcal{G}(\eta)$ has Weak Specification (at All Scales)

The goal of this section is to obtain Corollary 4.6, which shows that $\mathcal{G}(\eta)$ has weak specification at all scales.

Lemma 4.1 (Compare with Lemma 3.8 in [16]). Let $x \in S$ and β be a geodesic ray with $\beta(0) = x$. Then, for any $\varepsilon > 0$, there exist $T_0(\varepsilon)$ and a geodesic c which connects x with a point $z \in Con$ so that the length of c is at most $T_0(\varepsilon)$ and $\angle_x(\beta, c) < \varepsilon$ where $\angle_x(a, b)$ is the angle at x between geodesic segments a and b.

Proof.   Let $\tilde{x} \in \tilde{S}$ be a lift of x and $\tilde{\beta}$ a lift of β with $\tilde{\beta}(0) = \tilde{x}$. Denote by $C_{\frac{\varepsilon}{2}}(\tilde{x}, \tilde{\beta})$ the cone around β with vertex $\tilde{x}$ and angle $\frac{\varepsilon}{2}$. Choose $T_0 = T_0(\varepsilon)$ so large that an angle-ε sector of a radius-T_0 Euclidean ball contains a ball of radius much larger than the diameter of S. Then $I = B_{T_0}(\tilde{x}) \cap C_{\frac{\varepsilon}{2}}(\tilde{x}, \tilde{\beta})$ is at least as large as this Euclidean sector and so must contain a fundamental domain of S. Then $\widetilde{Con} \cap Int(I) \neq \emptyset$, so let $\tilde{z} \in \widetilde{Con} \cap Int(I)$ such that $\tilde{z}$ is closest to $\tilde{x}$. The segment $\tilde{c} = \tilde{x}\tilde{z}$ is a geodesic of length at most T_0. The projection of $\tilde{c}$ to S is the desired geodesic.   ∎

Lemma 4.2.   For any $\delta > 0$, there exists $T_1 = T_1(\delta)$ such that for any $t > 0$ and $(\gamma, t) \in \mathcal{G}(\eta)$, (γ, t) is δ-shadowed by a saddle connection path γ_e in the following sense:

- $\ell(\gamma_e) \leq t + 2T_1$;
- there exists $s_0 \in [0, T_1]$ with the property that if γ_e^c is any extension of γ_e to a complete geodesic then $d_{GS}(g_u(\gamma), g_u(g_{s_0}(\gamma_e^c))) \leq \delta$ for all $u \in [0, t]$.

In particular, if $t > \frac{\theta_0}{\eta}$, there exists a closed interval $I \supset [\frac{\theta_0}{2\eta}, t - \frac{\theta_0}{2\eta}]$ such that $\gamma_e(s_0 + u) = \gamma(u)$ for $u \in I$.

Proof.   As usual, we prove the result in $\tilde{S}$. Let $T = \max\{-\log(\delta), \frac{\theta_0}{2\eta}, \frac{\ell_0}{4}\}$ where θ_0 and ℓ_0 are from Lemma 2.15. By Lemma 2.12, if we construct $\tilde{\gamma}_e$ such that $d_{\tilde{S}}(\tilde{\gamma}(u), \tilde{\gamma}_e(s_0+u)) < \frac{\delta}{2}$ for all $u \in [-T, t + T]$, then $d_{GS}(g_u(\gamma), g_u(g_{s_0}(\gamma_e^c))) \leq \delta$ for all $u \in [0, t]$. (See Figure 3 for the constructions in this proof.)

By Proposition 3.9, there exist $t_0, t_1 \in [-\frac{\theta_0}{2\eta}, \frac{\theta_0}{2\eta}]$ such that $\tilde{\gamma}(t_0), \tilde{\gamma}(t + t_1) \in \widetilde{Con}$, $|\theta(\tilde{\gamma}, t_0)| - \pi \geq s\eta$ and $|\theta(\tilde{\gamma}, t+t_1)| - \pi \geq s\eta$. Thus, there exist $s_1 \in [-T, \frac{\theta_0}{2\eta}]$ and $s_2 \in [-\frac{\theta_0}{2\eta}, T]$ such that $\tilde{\gamma}(s_1), \tilde{\gamma}(t + s_2) \in \widetilde{Con}$ and $(\tilde{\gamma}([-T, s_1)) \cup \tilde{\gamma}((t+s_2, t + T])) \cap \widetilde{Con} = \emptyset$.
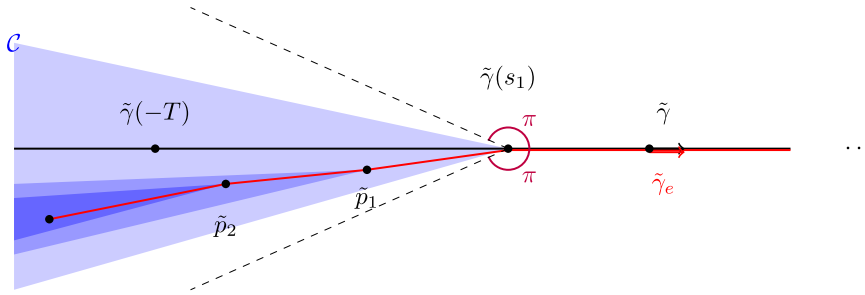
Fig. 3. The construction of γ_e in Lemma 4.2 around the left endpoint of γ. The sequence of α-cones featured in the proof is shaded.

If $s_1 = -T$, then define $\tilde{\gamma}_e(u - s_1) = \tilde{\gamma}(u)$ for $u \in [s_1, t + s_2]$. Assume $s_1 > -T$. Let η_0 be as in Lemma 2.15(b). Choose $\alpha < \frac{\ell_0}{4(T + \frac{\theta_0}{2\eta})} \min\{\eta_0, \frac{\delta}{2(\frac{\theta_0}{2\eta} + T)}\}$. Let $\mathcal{C}$ be the cone in $\tilde{S}$ around $\tilde{\gamma}([-T, s_1])$ with angle $\alpha/2$. Note that $\alpha < \eta_0$, so any geodesic segment from a point in $\mathcal{C}$ to $\tilde{\gamma}(s_1)$ can be concatenated with $\tilde{\gamma}([s_1, t])$ to form a geodesic. By Lemma 4.1, there exists $T_0 = T_0(\frac{\alpha}{2}) \geq T + \frac{\theta_0}{2\eta}$ and a point $\tilde{p}_1$ in $\widetilde{Con} \cap \mathcal{C}$ such that $d_{\tilde{S}}(\tilde{p}_1, \tilde{\gamma}(s_1)) \leq T_0$. Choose $\tilde{p}_1$ as in the previous sentence minimizing the distance to $\tilde{\gamma}([-T_0, s_1])$. If $d_{\tilde{S}}(\tilde{p}_1, \tilde{\gamma}(s_1)) \geq T + \frac{\theta_0}{2\eta}$, then let the initial segment of $\tilde{\gamma}_e$ be the geodesic segment $[\tilde{p}_1, \tilde{\gamma}(s_1)]$.

Otherwise, we repeat the argument above, applying Lemma 4.1 to construct an angle-$\alpha/2$ cone centered around the geodesic segment making angle $\pi + \frac{\alpha}{2}$ with $[\tilde{p}_1, \tilde{\gamma}(s_1)]$. We get a point $\tilde{p}_2 \in \widetilde{Con}$ in this cone with $d_{\tilde{S}}(\tilde{p}_1, \tilde{p}_2) \leq T_0$, again chosen to minimize the distance to $\tilde{\gamma}([-T_0, s_1])$. If $d_{\tilde{S}}(\tilde{p}_2, \tilde{\gamma}(s_1)) \geq T + \frac{\theta_0}{2\eta}$, then let the initial segment of $\tilde{\gamma}_e$ be the concatenation of geodesic segments $[\tilde{p}_2, \tilde{p}_1]$ and $[\tilde{p}_1, \tilde{\gamma}(s_1)]$. This concatenation is a geodesic by the choice of α and the construction of the cone. Otherwise, repeat the procedure at $\tilde{p}_2$ and so on.

We will need to repeat this procedure at most $\frac{T + \frac{\theta_0}{2\eta}}{\ell_0}$ times. We extend the beginning of $\tilde{\gamma}_e$ constructed here with $[\tilde{\gamma}(s_1), \tilde{\gamma}(t + s_2)]$ and then extend beyond $\tilde{\gamma}(t + s_2)$ (if needed) similarly to the procedure at $\tilde{\gamma}(s_1)$. Since the turning angles at each cone point are at least π, we obtain a saddle connection path $\tilde{\gamma}_e$.

Let $T_1 = T + \frac{\theta_0}{2\eta} + T_0$. Let s_0 be such that $\tilde{\gamma}_e(s_0 + s_1) = \tilde{\gamma}(s_1)$. Then $s_0 \in [0, T_1]$. For $u \in [s_1, s_2]$, $d_{\tilde{S}}(\tilde{\gamma}(u), \tilde{\gamma}_e(s_0 + u)) = 0 < \frac{\delta}{2}$, as desired. For $u \in [-T, s_1]$, note that the sequence of cones used in the proof have angles $\frac{\alpha}{2}$ and $\alpha < \frac{\ell_0}{4(T + \frac{\theta_0}{2\eta})} \frac{\delta}{2(\frac{\theta_0}{2\eta} + T)}$. There are at most $\frac{T + \frac{\theta_0}{2\eta}}{\ell_0}$ of these cones, each segment from $\tilde{p}_i$ to $\tilde{p}_{i+1}$ is at most length $T + \frac{\theta_0}{2\eta}$, and we always choose our cone points $\tilde{p}_i$ as close to $\tilde{\gamma}$ as we can. Therefore, the distance

$d_{\tilde{S}}(\tilde{\gamma}(u), \tilde{\gamma}_s(s_0 + u))$ is bounded by $\frac{\delta}{2}$ for $u \in [-T, s_1]$. For the same reason, this bound also holds for $u \in [s_2, t + T]$, finishing the proof. ∎

Lemma 4.3 (Compare with Lemma 3.9 in [16]). Let $N = [\frac{4\pi}{\eta_0}] + 3$ where η_0 is from Lemma 2.15(c). Let $q \in Con$. Then there exist N saddle connections $\sigma_1, \sigma_2, \dots, \sigma_N$ emanating from q with the following property.

For any geodesic segment γ with endpoint q, the concatenation of γ with at least one σ_i is also a local geodesic.

Proof. We have $\mathcal{L}(q) = 2\pi + \alpha \geq 2\pi + \eta_0$. Divide the space of directions at q into intervals of size no more than $\frac{\alpha}{2}$; at most $\lceil \frac{2\pi + \alpha}{\alpha/2} \rceil \leq N$ intervals are needed. Using Lemma 4.1, pick a saddle connection emanating from q with direction in each of these intervals. These are the σ_i.

The concatenation of γ and some saddle connection σ_i is a geodesic if and only if c_i lies outside of the π-cone of directions at q around γ. The complement of this cone in the space of directions at q is an interval of size $\mathcal{L}(q) - 2\pi = \alpha$ and must therefore fully contain one of our $\frac{\alpha}{2}$-size intervals. The σ_i chosen in this interval geodesically continues γ as desired. ∎

Lemma 4.4 (Compare with Corollary 3.1 in [16]). For any two parametrized saddle connections σ, σ' on S, there exists a geodesic segment γ that first passes through σ and eventually passes through σ'.

Proof. Let α be a closed geodesic that turns with angle greater than π at a cone point p (such α exists by Lemma 2.20). Denote by $\tilde{\sigma}$ the lift of σ to $\tilde{S}$ that has the starting point $\tilde{a}$ and the endpoint $\tilde{b}$. Consider a parametrized complete lift $\tilde{\alpha}$ of α such that it is disjoint from $\tilde{\sigma}$ and its positive endpoint is contained in the complement of the cone around $[\tilde{a}, \tilde{b}]$ with vertex $\tilde{b}$ and angle π. Denote by $\tilde{c}_t : [0, \ell_t] \to \tilde{S}$ the geodesic that connects $\tilde{a}$ with $\tilde{\alpha}(t)$. By the choice of the lift $\tilde{\alpha}$, there is a time t_0 such that for all $t \geq t_0$, $\tilde{c}_t$ passes through $\tilde{b}$ and that $\tilde{c}_{t_0}$ only shares its endpoint with $\tilde{\alpha}$. ∎

We now need the following fact.

Sublemma. There exists $t_1 > t_0$ such that $\tilde{c}_{t_1}$ intersects the geodesic segment $[\tilde{\alpha}(t_0), \tilde{\alpha}(t)]$ in a positive-length segment.

Proof of Sublemma.   Consider the geodesic triangle in $\tilde{S}$ with vertices $\tilde{a}$, $\tilde{\alpha}(t_0)$ and $\tilde{\alpha}(t)$ for $t > t_0$. As t increases, the length of the side $[\tilde{\alpha}(t_0), \tilde{\alpha}(t)]$ increases without bound while the length of $[\tilde{a}, \tilde{c}(t_0)]$ is fixed, so the length of $\tilde{c}_t = [\tilde{a}, \tilde{\alpha}(t)]$ must also increase without bound once t is sufficiently large, by the triangle inequality. The comparison triangles in $\mathbb{R}^2$ will have one side of fixed length while the other two become very long. The angle at the vertex of the comparison triangle where the long sides meet must therefore become arbitrarily small.

At each lift of the cone point p that α passes through, $\tilde{\alpha}$ has turning angle $\pi + \theta$ for some $\theta > 0$. Let T be so large that the angle noted above in the Euclidean comparison triangle is $< \theta$. As $\tilde{S}$ is CAT(0), the original triangle in $\tilde{S}$ has angles no larger than those in the comparison triangle. Thus, the angle between $\tilde{c}_t$ and $[\tilde{\alpha}(t_0), \tilde{\alpha}(t)]$ will be less than θ for all $t \geq T$. Let t' be any time greater than T at which $\tilde{\alpha}$ passes through a lift of the cone point, and let $t_1 > t'$. Since $\tilde{\alpha}$ turns with excess angle θ at $\tilde{\alpha}(t')$, the concatenation of $\tilde{c}(t')$ and $\tilde{\alpha}([t', \infty))$ is a geodesic ray. Therefore, $\tilde{c}_{t_1}$ and $[\tilde{\alpha}(t_0), \tilde{\alpha}(t)]$ intersect in a positive-length segment. ∎

For t_1 as in the sublemma, the projection of $\tilde{c}_{t_1}$ to S is a local geodesic that first passes through σ and eventually through a piece of α. By extending the resulting local geodesic along α, we can make sure that it passes through the whole curve α. We denote the resulting local geodesic by g_1.

We apply the above argument to σ' and α with their orientations reversed to obtain a local geodesic g_2 that connects these curves.

The concatenation of g_1 and g_2 (with its orientation reversed) has the desired property.

Repeating the proof of Proposition 3.2 in [16] and replacing [16, Lemma 3.9] by Lemma 4.3 and [16, Corollary 3.1] by Lemma 4.4, we obtain Proposition 4.5 that strengthens Lemma 4.4. We include the proof of the proposition for completeness.

Proposition 4.5.   (Compare with Proposition 3.2 in [16]) There exists a constant $C(S) > 0$ so that the following holds.

For any two parametrized saddle connections σ, σ' on S, there exists a geodesic segment γ that first passes through σ and eventually passes through σ' and that is of length at most $C(S) + \ell(\sigma) + \ell(\sigma')$.

Proof.   Recall that S has only finitely many cone points. By Lemma 4.3, there are $N_0 = N_0(S)$ parametrized saddle connections $\sigma_1, \sigma_2, \ldots, \sigma_{N_0}$ with the property that for

any geodesic segment with endpoint in *Con* (in particular, any saddle connection) the concatenation of it with at least one σ_i is a local geodesic. By Lemma 4.4, for each pair (σ_i, σ_j), there is a local geodesic c_{ij} that first passes through σ_i and eventually through σ_j. Since there are only finitely many pairs (σ_i, σ_j), there exists a constant $C(S)$ such that $\ell(c_{ij}) \leq C(S)$. Thus, for any two parametrized saddle connections σ, σ', we do the following. First, we connect the endpoint of σ to σ_i for some i and the starting point of σ' (the endpoint of the saddle connection with the reversed parametrization of σ') to σ_j for some j so that the results of concatenations are local geodesics. Then, the concatenation of σ with c_{ij} followed by the concatenation with σ' is the desired geodesic segment that first passes through σ and eventually through σ' of length at most $C(S) + \ell(\sigma) + \ell(\sigma')$. ∎

Using Lemma 4.2 and Proposition 4.5, we obtain the weak specification property on $\mathcal{G}(\eta)$ at all scales.

Corollary 4.6. (Weak specification) For all $\delta > 0$, there exists $T = T(\eta, \delta, S) > 0$ such that for all $(\gamma_1, t_1), \ldots, (\gamma_k, t_k) \in \mathcal{G}(\eta)$ there exist $0 = s_1 < s_2 < \ldots < s_k$ and a geodesic γ on S such that for all $i = 1, \ldots, k$ we have $s_{i+1} - (s_i + t_i) \in [0, T]$ and $d_{GS}(g_u(\gamma_i), g_u(g_{s_i}(\gamma))) < \delta$ for all $u \in [0, t_i]$.

Proof. We can take $T = 2T_1 + C(S)$ where T_1 is as in Lemma 4.2 and $C(S)$ is as in Proposition 4.5. We omit the proof here as it is a simplified version of the proof of Proposition 5.6. ∎

5 $\mathcal{G}(\eta)$ has Strong Specification (at All Scales)

The goal of this section is to upgrade the weak specification property of Corollary 4.6 to strong specification (Proposition 5.6), in which we have more precise control over when our shadowing geodesic shadows each segment.

As η is fixed throughout, we write $\mathcal{G} := \mathcal{G}(\eta)$.

Lemma 5.1. If $G \subset \mathbb{R}^{\geq 0} \not\subset c\mathbb{N}$ for all $c > 0$, then for all $\delta > 0$, there exist $x, y \in G$ and $n, m \in \mathbb{N}$ such that $0 < nx - my < \delta$.

Proof. Let x denote the smallest nonzero element of G, which exists, as otherwise we are immediately done. Now, there are three cases.

First, assume there exists $y \in G$ such that $\frac{y}{x} \notin \mathbb{Q}$. Now take $q \in \mathbb{N}$ large enough so that $\frac{x}{q} < \delta$, and so that there is $p \in \mathbb{N}$ with $|\frac{y}{x} - \frac{p}{q}| < \frac{1}{q^2}$ by Dirichlet's theorem. Then, this implies that

$$|qy - px| < \frac{x}{q} < \delta.$$

In the 2nd case, suppose that for all $y \in G$, $\frac{y}{x}$ is rational, and when written in lowest terms, the denominators can be arbitrarily large. Then, take n such that $\frac{x}{n} < \delta$ and $y \in G$ with $\frac{y}{x} = \frac{p}{q}$ in lowest terms for some $q > n$. Then, as p is invertible in $\mathbb{Z}/q\mathbb{Z}$, we can take m to be a positive integer such that $mp \equiv 1 \pmod{q}$. It follows that

$$\left| \frac{mp - 1}{q} x - my \right| = \frac{x}{q} < \delta.$$

Finally, in the 3rd case, $\frac{y}{x}$ is always rational, but with denominators bounded above by M. Then, $G \subset \frac{x}{M!}\mathbb{N}$, a contradiction. ∎

Lemma 5.2.   Suppose $x > y > 0$ and $x - y = \delta$. Then, there exists $T > 0$ such that for all $\tau \geq T$ and all $n \in \mathbb{N} \cup \{0\}$, there exists $m_1, m_2 \in \mathbb{N}$ such that $\tau + n\delta \leq m_1 x + m_2 y \leq \tau + (n+1)\delta$.

Proof.   Fix C such that $C > \frac{y}{\delta} + 2$. We claim that $T = \max\{Cy, 1\}$. Fix $\tau \geq T$. Now, let $n \in \mathbb{N} \cup \{0\}$. Fix k_1 to be the largest integer such that $k_1 y \leq \tau + n\delta$ and then choose k_2 to be the smallest positive integer such that $k_1 y + k_2 \delta \geq \tau + n\delta$. Therefore, we see that $k_2 x + (k_1 - k_2)y = k_1 y + k_2 \delta$, and so

$$\tau + n\delta \leq k_2 x + (k_1 - k_2)y \leq \tau + (n+1)\delta.$$

Observe that by construction,

$$k_1 y + (k_2 - 1)\delta < \tau + n\delta < k_1 y + y,$$

and consequently, $k_2 < \frac{y}{\delta} + 1$. Therefore, by our choices of τ and C,

$$k_1 > \frac{\tau + n\delta - y}{y} > \frac{Cy - y}{y} > \frac{y}{\delta} + 1.$$

Thus, $k_1 - k_2 > 0$, and we are done. ∎

We need the following result of Ricks; we explain the necessary terminology in the course of applying it.

Theorem 5.3. [19, Theorems 4 and 5]   Let X be a proper, geodesically complete, CAT(0) space under a proper, cocompact, isometric action by a group Γ with a rank one element, and suppose X is not isometric to the real line. Then, the length spectrum is arithmetic if and only if there is some $c > 0$ such that X is isometric to a tree with all edge lengths in $c\mathbb{Z}$.

Proposition 5.4.   Given $\delta > 0$, there exist two closed saddle connection paths γ, ξ such that $0 < |\ell(\gamma) - \ell(\xi)| < \delta$.

Proof.   This follows for translation surfaces by combining Lemma 5.1 with Section 6 of [12] (see hypothesis (T3) and the discussion following [12, Proposition 6.9]).

For general flat surfaces with conical points, this follows from Theorem 5.3. We outline the reasoning as follows. We say that $\gamma \in \Gamma$ is rank one if there exists a geodesic η such that $\gamma \eta = g_t \eta$ for some $t > 0$ and η does not bound a flat half plane, that is, a subspace isometric to $\mathbb{R} \times [0, \infty)$. The existence of this follows from the existence of a closed geodesic which turns with angle greater than π at some cone point (see Lemma 2.20). Now, the universal cover of a flat surface with cone points is not isometric to a tree with edge lengths in $c\mathbb{Z}$, and so it follows that the length spectrum is not arithmetic. The length spectrum is the collection of lengths of hyperbolic isometries in Γ, which is precisely the set of lengths of closed geodesics, which by Lemma 2.18 is the set of lengths of closed saddle connection paths. We can now apply Lemma 5.1.   ∎

Proposition 5.5.   For all $\delta > 0$, there exists $\tau = \tau(\delta) > 0$ and $\delta' < \delta$ such that for any $\tau' > \tau$, any two saddle connections σ, σ' and any $n \in \mathbb{N} \cup \{0\}$, there exists a geodesic segment ξ_n that begins with σ and ends with σ' with length in $[\ell(\sigma) + \ell(\sigma') + \tau' + n\delta', \ell(\sigma) + \ell(\sigma') + \tau' + (n+1)\delta']$.

Proof.   Fix $\delta > 0$, and take γ_1, γ_2 to be closed geodesics such that $0 < |\ell(\gamma_1) - \ell(\gamma_2)| = \delta' < \delta$, which exist by Proposition 5.4. Now take $\tau = 3C(S) + T$, where $C(S)$ is from Proposition 4.5 and T is from Lemma 5.2 applied for $\ell(\gamma_1)$ and $\ell(\gamma_2)$.

Consider two saddle connections σ and σ', and apply Proposition 4.5 three times to connect, in sequence, σ to γ_1 to γ_2 to σ' with the geodesic ξ. Furthermore, $\ell(\xi) = L + \ell(\sigma) + \ell(\gamma_1) + \ell(\gamma_2) + \ell(\sigma')$ and $L \leq 3C(S)$. Because the γ_i are closed geodesics,

there is a geodesic ξ_{k_1,k_2} that follows the exact path of ξ except that it loops around γ_i a total of k_i times. In other words, $\ell(\xi_{k_1,k_2}) = \ell(\xi) + (k_1 - 1)\ell(\gamma_1) + (k_2 - 1)\ell(\gamma_2)$. Now let $n \in \mathbb{N}$, and, using Lemma 5.2, take k_1, k_2 such that

$$k_1 \ell(\gamma_1) + k_2 \ell(\gamma_2) \in [T + (3C(S) - L) + (\tau' - \tau) + n\delta', T + (3C(S) - L) + (\tau' - \tau) + (n+1)\delta'].$$

Then $\xi_n := \xi_{k_1,k_2}$ satisfies our desired property.   ∎

Proposition 5.6.   The collection of orbit segments $\mathcal{G} = \mathcal{G}(\eta)$ has strong specification at all scales. That is, for any $\varepsilon > 0$, there exists $\hat{\tau}(\varepsilon) > 0$ such that for any finite collection $\{(\gamma_i, t_i)\}_{i=1}^{n} \subset \mathcal{G}$, there exists $\hat{\xi} \in GS$ that ε-shadows the collection with transition time $\hat{\tau}$ between orbit segments. In other words, for $1 \leq i \leq n$,

$$d_{GS}(g_{u+\sum_{j=1}^{i-1}(t_j+\hat{\tau})}\hat{\xi}, g_u \gamma_i) \leq \varepsilon \text{ for } 0 \leq u \leq t_i.$$

Moreover, for $1 \leq i \leq n$ such that $t_i > \frac{\theta_0}{\eta}$ where θ_0 as in Lemma 2.15(d), there exists a closed interval $I_i \supset [\frac{\theta_0}{2\eta}, t_i - \frac{\theta_0}{2\eta}]$ such that $\hat{\xi}(u + \sum_{j=1}^{i-1}(t_j + \hat{\tau})) = \gamma_i(u)$ for $u \in I_i$.

Proof.   By Lemma 4.2, there exists $T_1 = T_1(\frac{\varepsilon}{2}, S)$ for each $i = 1, \ldots, n$, there exists a saddle connection path γ_i^e such that $\ell(\gamma_i^e) \leq t_i + 2T_1$ and there exists $s_i \in [0, T_1]$ such that for any extension $\hat{\gamma}_i^e$ of γ_i^e to a complete geodesic, we have

$$d_{GS}(g_u(\gamma_i), g_u(g_{s_i}(\hat{\gamma}_i^e))) \leq \frac{\varepsilon}{2} \quad \text{for all} \quad u \in [0, t_i].$$

Moreover, if $t_i > \frac{\theta_0}{\eta}$, there exists a closed interval $I_i \supset [\frac{\theta_0}{2\eta}, t_i - \frac{\theta_0}{2\eta}]$ such that $\gamma_i^e(s_i + u) = \gamma_i(u)$ for $u \in I_i$. We will construct our shadowing geodesic by induction. Let $\tau = \tau\left(\frac{\varepsilon}{4}\right)$, $\delta' < \frac{\varepsilon}{4}$ be as in Proposition 5.5 applied for $\delta = \frac{\varepsilon}{4}$. Denote $T = \tau + 3T_1$.

Thus, for any $k = 1, \ldots, n-1$ and $m_k \in \mathbb{N} \cup \{0\}$, there exists a geodesic segment ξ_{k+1} that begins with γ_k^e and ends with γ_{k+1}^e with length $\ell(\xi_{k+1}) = \ell(\gamma_k^e) + \ell(\gamma_{k+1}^e) + T - (s_{k+1} - s_k) - (\ell(\gamma_k^e) - t_k) + c_k$ where $c_k \in [m_k \delta', (m_k + 1)\delta']$.

Moreover, by Lemma 4.2, for any extension $\hat{\xi}_{k+1}$ of ξ_{k+1} to a complete geodesic with $\hat{\xi}_{k+1}(u) = \xi_{k+1}(s_k + u)$ for all $u \in [-s_k, -s_k + \ell(\xi_{k+1})]$, we have

$$d_{GS}(g_u \hat{\xi}_k, g_u \gamma_k) \leq \frac{\varepsilon}{2} \quad \text{for all} \quad u \in [0, t_k] \quad \text{and}$$

$$d_{GS}(g_u(g_{t_k+T+c_k}\hat{\xi}_{k+1}), g_u \gamma_{k+1}) \leq \frac{\varepsilon}{2} \quad \text{for all} \quad u \in [0, t_{k+1}]. \tag{9}$$

We define the sequence m_k inductively. Let $m_1 = 0$. In particular, $c_1 \in [0, \delta'] \subset [0, \frac{\varepsilon}{4}]$. For $k > 1$, we set

$$m_k = \lceil \frac{\varepsilon}{4\delta'} \rceil \quad \text{if} \quad (k-1)\frac{\varepsilon}{4} - \sum_{i \leq k-1} c_i > \frac{\varepsilon}{4}, \quad \text{and} \quad 0 \quad \text{otherwise,}$$

as this ensures $\left| \sum_{j=1}^{k-1} \frac{\varepsilon}{4} - c_j \right| < \frac{\varepsilon}{2}$.

Let ξ be a geodesic segment that is a result of gluing ξ_k and ξ_{k+1} along γ_k^e that is the end of ξ_k and the beginning of ξ_{k+1} for all $k = 2, \ldots, n-1$. Let $\hat{\xi}$ be any extension of ξ to a complete geodesic with the parametrization such that $\hat{\xi}(-s_1) = \xi(0)$. By the choice of m_k and (9), we obtain for $1 \leq i \leq n$,

$$d_{GS}(g_u(g_{\sum_{j=1}^{i-1}(t_j+T+\varepsilon/4)}\hat{\xi}), g_u\gamma_i) \leq d_{GS}(g_u(g_{\sum_{j=1}^{i-1}(t_j+T+c_j)}\hat{\xi}), g_u\gamma_i) + \frac{\varepsilon}{2} \leq \varepsilon \quad \text{for all} \quad u \in [0, t_i].$$

Thus, $\hat{\xi}$ is the desired shadowing geodesic. As a result, the collection of orbit segments $\mathcal{G}$ has strong specification at all scales with the specification constant $T + \frac{\varepsilon}{4}$. ∎

We close this section by recording a simple technical modification of Proposition 5.6, which we will need when we apply specification in Section 7.

Definition 5.7.   Let $M > 0$ and $\eta > 0$ be given. We denote by $\mathcal{G}^M(\eta)$ the set of all orbit segments (γ, t) such that there exist t_1, t_2 with $|t_i| < M$ such that $(g_{t_1}\gamma, t - t_1 + t_2) \in \mathcal{G}(\eta)$. That is, these are segments that lie in $\mathcal{G}(\eta)$ after making some bounded change to their endpoints.

Corollary 5.8.   Specification as in Proposition 5.6 holds for $\mathcal{G}^M(\eta)$, with the constant T depending on M in addition to the parameters listed in Proposition 5.6.

Proof.   This is a simple exercise using Proposition 5.6 and uniform continuity of the geodesic flow. We give the idea of the proof. Let $\{(\gamma_i, t_i)\}_{i=1}^n \subset \mathcal{G}^M(\eta)$ be a collection of segments that we wish to shadow at scale ε. This leads to a collection $\{(g_{s_i}\gamma_i, t_i')\}_{i=1}^n \subset \mathcal{G}(\eta)$, where $|s_i| \leq M$ and $|t_i - t_i'| \leq M$ that we can shadow at any scale as in Proposition 5.6. We choose our new shadowing scale δ so that if $d_{GS}(\gamma, \xi) < \delta$, then $d_{GS}(g_t\gamma, g_t\xi) < \varepsilon$ for $t \in [-M, M]$, using uniform continuity of the flow. Any geodesic that δ-shadows $\{(g_{s_i}\gamma_{t_i}, t_i')\}$ must then ε-shadow our desired collection $\{(\gamma_i, t_i)\}$. ∎

6 $\mathcal{G}(\eta)$ **has the Bowen Property**

In this section, we establish the Bowen property (see Definition 2.9). To do so, we analyze orbits that stay close to a good orbit segment for some time. This description will allow us to effectively bound the difference of ergodic averages along these orbits.

Proposition 6.1.   For all $\eta > 0$, for all sufficiently small $\varepsilon > 0$ (dependent on η), and for any $(\gamma, t) \in \mathcal{G}(\eta)$ with $t > 2\frac{\theta_0}{2\eta}$, we have

$$B_t(\gamma, \varepsilon) \subset C_{2\varepsilon, \frac{\theta_0}{2\eta}}(\gamma, t),$$

where

$$B_t(\gamma, \varepsilon) = \{\xi \in GS \mid d_{GS}(g_u\gamma, g_u\xi) < \varepsilon \text{ for all } u \in [0, t]\}$$

and

$$C_{2\varepsilon, \frac{\theta_0}{2\eta}}(\gamma, t) = \left\{\xi \ \Big| \ \exists |r| \leq 2\varepsilon \text{ such that } g_r\xi(u) = \gamma(u) \text{ for all } u \in \left[\frac{\theta_0}{2\eta}, t - \frac{\theta_0}{2\eta}\right]\right\}.$$

Proof.   Fix $\eta > 0$, and recall Proposition 3.9. Now choose $\varepsilon > 0$ small enough that $s\sin(\frac{s\eta}{4}) > 2\varepsilon e^{2(\frac{\theta_0}{2\eta}+s)}$. (Here, s is the parameter involved in the definition of λ and fixed in Lemma 3.8.) Consider a cone around some geodesic with angle $\frac{s\eta}{4}$. By an easy computation, the ball of radius $2\varepsilon e^{2(\frac{\theta_0}{2\eta}+s)}$ with center at distance s from the cone point along the geodesic is contained in the cone (recall that $s > 0$ was chosen so that $2s < \ell_0$).

   Let $(\gamma_1, t) \in \mathcal{G}(\eta)$ with $t > \frac{\theta_0}{\eta}$ be arbitrary. By Proposition 3.9, there exists $t_0 \in [-\frac{\theta_0}{2\eta}, \frac{\theta_0}{2\eta}]$ such that $\gamma_1(t_0) \in Con$ and $|\theta(\gamma_1, t_0)| - \pi \geq s\eta$. Similarly, there exists $t_1 \in [-\frac{\theta_0}{2\eta}, \frac{\theta_0}{2\eta}]$ such that $\gamma_1(t + t_1) \in Con$ and $|\theta(\gamma_1, t + t_1)| - \pi \geq s\eta$.

   Now consider $\gamma_2 \in B_t(\gamma_1, \varepsilon)$. Taking t_0 and t_1 as above, by Lemmas 2.11 and 2.14,

$$d_S(\gamma_1(t_0 - s), \gamma_2(t_0 - s)) \leq 2d_{GS}(g_{t_0-s}\gamma_1, g_{t_0-s}\gamma_2) \leq 2d_{GS}(\gamma_1, \gamma_2)e^{2|\frac{\theta_0}{2\eta}-s|} \leq 2\varepsilon e^{2(\frac{\theta_0}{2\eta}+s)} \quad (10)$$

and

$$d_S(\gamma_1(t+t_1+s), \gamma_2(t+t_1+s)) \leq 2d_{GS}(g_{t_1+s}g_t\gamma_1, g_{t_1+s}g_t\gamma_2) \leq 2d_{GS}(g_t\gamma_1, g_t\gamma_2)e^{2|t_1+s|} \leq 2\varepsilon e^{2(\frac{\theta_0}{2\eta}+s)}. \quad (11)$$

   Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be lifts of γ_1 and γ_2 to $\widetilde{GS}$ so that $d_{\widetilde{GS}}(\tilde{\gamma}_1, \tilde{\gamma}_2) = d_{GS}(\gamma_1, \gamma_2)$. Let $B_1 = B(\tilde{\gamma}_1(t_0 - s), 2\varepsilon e^{2(\frac{\theta_0}{2\eta}+s)})$ and $B_2 = B(\tilde{\gamma}_1(t + t_1 + s), 2\varepsilon e^{2(\frac{\theta_0}{2\eta}+s)})$. Then, by (10) and

(11) and the remarks in the 1st paragraph of this proof, γ_2 intersects B_1 and B_2. Since $|\theta(\gamma_1, t_0)| - \pi \geq s\eta$ and $|\theta(\gamma_1, t + t_1)| - \pi \geq s\eta$, by the choice of ε and the fact that any two points in a CAT(0)-space are connected by a unique geodesic segment, $\tilde{\gamma}_2$ contains $\tilde{\gamma}_1[t_0, t + t_1]$. Moreover, since $d_{G\tilde{S}}(g_{t_0+s}\tilde{\gamma}_1, g_{t_0+s}\tilde{\gamma}_2) \leq \varepsilon$ and $0 \leq t_0 + s \leq 2s < t$, it follows that $d_S(\tilde{\gamma}_1(t_0+s), \tilde{\gamma}_2(t_0+s)) \leq 2\varepsilon$. Thus, there exists r such that $|r| \leq 2\varepsilon$ and $g_r\gamma_2(u) = \gamma_1(u)$ for $u \in [t_0, t + t_1]$. Since $t_0 \leq \frac{\theta_0}{2\eta}$ and $t_1 \geq -\frac{\theta_0}{2\eta}$, we have completed our proof. ∎

Proposition 6.2.   For all $\varepsilon, s > 0$ and α-Hölder continuous functions ϕ, there exists $K > 0$ such that for all geodesic segments (γ_1, t) with $t > 2\frac{\theta_0}{2\eta}$, given any $\gamma_2 \in C_{2\varepsilon,s}(\gamma_1, t)$, we have

$$\left| \int_0^t \phi(g_r\gamma_1) - \phi(g_r\gamma_2) \, dr \right| \leq K.$$

Proof.   Let R be the time-shift in the definition of $C_{2\varepsilon,s}(\gamma_1, t)$, so that $g_R\gamma_2(r) = \gamma_1(r)$ for $r \in [s, t - s]$. We see that

$$\left| \int_0^t \phi(g_r\gamma_1) - \phi(g_r\gamma_2) \, dr \right| \leq \left| \int_0^t \phi(g_r\gamma_1) \, dr - \int_{-R}^{t-R} \phi(g_r(g_R\gamma_2)) \, dr \right|$$

$$\leq \left| \int_s^{t-s} \phi(g_r\gamma_1) - \phi(g_r(g_R\gamma_2)) \, dr \right| + (4s + 2|R|)\|\phi\|.$$

Since $\gamma_1 = g_R\gamma_2$ on $[s, t - s]$, by Lemma 2.13, we have for all $r \in [s, t - s]$,

$$d_{GS}(g_r\gamma_1, g_r(g_R)\gamma_2) \leq e^{-2\min\{|r-s|, |r-(t-s)|\}}.$$

Thus, we obtain

$$\left| \int_s^{t-s} \phi(g_r\gamma_1) - \phi(g_r(g_R\gamma_2)) \, dr \right| \leq \int_s^{t-s} C(d_{GS}(g_r\gamma_1, g_r(g_R\gamma_2)))^\alpha \, dr$$

$$\leq \int_s^{\frac{t}{2}} C e^{-2\alpha(r-s)} \, dr + \int_{\frac{t}{2}}^{t-s} C e^{-2\alpha((t-s)-r)} \, dr$$

$$= \frac{C}{\alpha}(1 - e^{-\alpha(t-2s)})$$

$$\leq \frac{C}{\alpha}.$$

As a result, since $|R| < 2\varepsilon$, we have

$$\left| \int_0^t \phi(g_r\gamma_1) - \phi(g_r\gamma_2) \, \mathrm{d}r \right| \leq \frac{C}{\alpha} + (4s + 4\varepsilon)\|\phi\|.$$

∎

Corollary 6.3.   For all $\eta > 0$, there exists $\varepsilon > 0$ such that $\mathcal{G}(\eta)$ has the Bowen property at scale ε.

Proof.   Fix $\eta > 0$. Then, choose $\varepsilon > 0$ sufficiently small to apply the previous propositions. Then, we can take the constant for the Bowen property to be $\max\{K, 2\frac{\theta_0}{\eta}\|\phi\|\}$, where K is from the previous proposition. Then, the previous proposition gives the desired bound for orbit segments of length at least $\frac{\theta_0}{\eta}$, and the triangle inequality gives the desired bound for any shorter orbit segments.   ∎

7   Establishing the Pressure Gap

In this section, we prove the pressure gap condition of [4] for certain potentials. We then show that this pressure gap holds in the product space as well. See also the survey by Climenhaga and Thompson [11, Section 14].

First, we prove the following theorem.

Theorem 7.1.   Let ϕ be a continuous potential that is locally constant on a neighborhood of Sing. Then, $P(\mathrm{Sing}, \phi) < P(\phi)$.

Furthermore, we use the above theorem to note that a pressure gap also holds for functions that are nearly constant. (See Corollary 7.8.) For a sense of the functions covered by Theorem 7.1, it may be helpful to think of the special case of a translation surface. There are infinitely many cylinders in such an S, and the geodesics circling different cylinders are in different connected components of Sing, so there is significant flexibility in building a function that satisfies Theorem 7.1 on Sing itself, let alone on the complement of its neighborhood.

Our argument for Theorem 7.1 closely follows that in Section 8 of [4]. The different geometry in our situation calls for somewhat different arguments in Proposition 7.4 and Lemma 7.5, which we present here in full. After these are proved, the argument hews closely to [4]. We present the main steps of the argument, filling in the details where a modification is necessary for the present situation.

For any $\eta > 0$, we let

$$\mathrm{Reg}(\eta) = \{\gamma \mid \lambda(\gamma) \geq \eta\}.$$

We need a pair of lemmas in this section.

Lemma 7.2.   Let c be any singular geodesic segment. That is, c is a geodesic segment such that the turning angle at any cone points it encounters is always $\pm\pi$. Then c can be extended to a complete geodesic $\gamma \in \mathrm{Sing}$.

Proof.   The extension is accomplished by following the geodesic trajectory established by c and, whenever a cone point is encountered, continuing the extension so that a turning angle of π or $-\pi$ is made.   ∎

Let $\partial_\infty \tilde{S}$ be the boundary at infinity of $\tilde{S}$, equipped with the usual cone topology (see, e.g., [3, Section II.8]). Since S is a surface, $\partial_\infty \tilde{S}$ is a circle. Using this identification, we can speak of a path in $\partial_\infty \tilde{S}$ as being monotonic if it always moves in a clockwise or counterclockwise direction.

The following lemma leverages this structure to provide a way to continuously move a geodesic in $G\tilde{S}$.

Lemma 7.3.   Let $\tilde{\gamma} \in G\tilde{S}$ with $\tilde{\gamma}(t_0) = \xi \in \widetilde{Con}$. Let ζ_v be a continuous and monotonic path in $\partial_\infty \tilde{S}$ with $\zeta_0 = \tilde{\gamma}(+\infty)$ such that for all v, the ray connecting ξ with ζ_v can be concatenated with $\tilde{\gamma}(-\infty, t_0)$ to form a geodesic $\tilde{\gamma}_v$. Then $\{\gamma_v\}$ is a continuous path of geodesics in $G\tilde{S}$ with $d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\gamma}_v)$ nondecreasing in $|v|$.

Proof.   First, that ξ and ζ_v can be connected with a unique geodesic ray is a standard fact about CAT(0) spaces [3, Section II.8, Prop. 8.2]. For continuity of $\tilde{\gamma}_v$, we claim that if $v \to v_0$, $d_{\tilde{S}}(\tilde{\gamma}_v(t), \tilde{\gamma}_{v_0}(t)) \to 0$ uniformly on any $[t_0, T]$. This together with the formula for $d_{G\tilde{S}}$ will show that $d_{G\tilde{S}}(\gamma_v, \gamma_{v_0}) \to 0$. To verify the claim, fix $T > t_0$ and $\varepsilon > 0$ and recall that in the cone topology on $\partial_\infty \tilde{S}$,

$$U(\tilde{\gamma}_{v_0}, T, \varepsilon) := \{\zeta \in \partial_\infty \tilde{S} : d_{\tilde{S}}(c(T), \tilde{\gamma}_{v_0}(T)) < \varepsilon \text{ where } c \text{ is the geodesic ray from } \xi \text{ to } \zeta\}$$

is a basic open set around $\zeta_{v_0} = \tilde{\gamma}_{v_0}(+\infty)$ [3, Section II.8]. Therefore, for v sufficiently close to v_0, $\zeta_v \in U(\tilde{\gamma}_{v_0}, T, \varepsilon)$. But the ray c from ξ to ζ_v is precisely $\tilde{\gamma}_v|_{[t_0, +\infty)}$. Thus, $d_{\tilde{S}}(\tilde{\gamma}_v(T), \tilde{\gamma}_{v_0}(T)) < \varepsilon$. Since the distance between two geodesics is a convex function

of the parameter [3, Section II.2] and $d_{\tilde{S}}(\tilde{\gamma}_v(t_0), \tilde{\gamma}_{v_0}(t_0)) = 0$, for all $t \in [t_0, T]$, we have $d_{\tilde{S}}(\tilde{\gamma}_v(t), \tilde{\gamma}_{v_0}(t)) < \varepsilon$ and hence have the desired uniform convergence.

For all $t \leq t_0$, $d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\gamma}_v(t)) = 0$. We claim that for $t > t_0$, $d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\gamma}_v(t))$ is nondecreasing in $|v|$. Together with the formula for $d_{G\tilde{S}}$, this will provide the result.

Fix some $v^* \neq 0$; without loss of generality, we can assume $v^* > 0$. Since ζ_v is a monotonic path on $\partial_\infty \tilde{S}$, $v \mapsto \tilde{\gamma}_v(t)$ sweeps out an arc on the circle of radius $t - t_0$ centered at ξ monotonically (though not necessarily strictly monotonically). We want to show that for $v > v^*$, $d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\gamma}_v(t)) \geq d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\gamma}_{v^*}(t))$. This will be trivially true if for all $v > v^*$, $\tilde{\gamma}_v(t) = \tilde{\gamma}_{v^*}(t)$, so we can assume this is not the case.

Consider the path swept out by $v \mapsto \tilde{\gamma}_v(t)$. Near the point $\tilde{\gamma}_{v^*}(t)$ this path consists of arcs of two Euclidean circles meeting at $\tilde{\gamma}_{v^*}(t)$. To each side of $\tilde{\gamma}_{v^*}(t)$, the arc belongs to a circle centered at the cone point on $[\xi, \tilde{\gamma}_{v^*}(t)] \setminus \{\tilde{\gamma}_{v^*}(t)\}$ closest to $\tilde{\gamma}_{v^*}(t)$ among those cone points where $[\xi, \tilde{\gamma}_{v^*}(t)]$ makes angle greater than π on the given side of $[\xi, \tilde{\gamma}_{v^*}(t)]$. Therefore, in the space of directions at $\tilde{\gamma}_{v^*}(t)$ (this will be the tangent space at $\tilde{\gamma}_{v^*}(t)$ unless $\tilde{\gamma}_{v^*}(t)$ happens to be a cone point), we have well-defined vectors pointing along these arcs. Furthermore, since these are arcs of Euclidean circles, the angles between these two vectors and a vector pointing radially along $[\tilde{\gamma}_{v^*}(t), \xi]$ are both $\frac{\pi}{2}$. Let W^+ and W^- be vectors in the space of directions at $\tilde{\gamma}_{v^*}(t)$ pointing along the arc swept out by $v \mapsto \tilde{\gamma}_v(t)$ with W^+ pointing in the direction swept out as v increases past v^* and W^- in the direction swept out as v decreases from v^*. (Note that $v \mapsto \tilde{\gamma}_v(t)$ may be constant in v for v near v^* due to cone points $\tilde{\gamma}_{v^*}$ encounters at times greater than t. The vectors $W^\pm$ are tangent to a reparametrization of this curve by arc-length, for instance.) Similarly, let $H^\pm$ be the vectors in the space of directions at $\tilde{\gamma}_{v^*}(t)$ pointing along the circle of radius $d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\gamma}_{v^*}(t))$ centered at $\tilde{\gamma}(t)$. Let V_1 be the initial tangent vector for the geodesic segment from $\tilde{\gamma}_{v^*}(t)$ to ξ, and let V_2 be the initial tangent vector for the geodesic segment from $\tilde{\gamma}_{v^*}(t)$ to $\tilde{\gamma}(t)$. By the CAT(0) condition and using a comparison triangle for the triangle with vertices $\xi, \tilde{\gamma}(t)$, and $\tilde{\gamma}_{v^*}(t)$, it is easy to check that the angle between V_1 and V_2 is in $[0, \frac{\pi}{2})$. The angles between $W^\pm$ and V_1 and between $H^\pm$ and V_2 are all $\frac{\pi}{2}$ as these are angles between a circle and one of its radial segments. (See Figure 4.)

The segment $[\tilde{\gamma}_{v^*}(t), \tilde{\gamma}(t)]$ lies in the convex hull of $\tilde{\gamma}$ and $\tilde{\gamma}_{v^*}$. By the CAT(0) condition, it is within the ball of radius $t - t_0$ centered at ξ. So V_2, which points along $[\tilde{\gamma}_{v^*}(t), \tilde{\gamma}(t)]$, is between V_1 and W^- in the space of directions at $\tilde{\gamma}_{v^*}(t)$. More precisely, the space of directions at $\tilde{\gamma}_{v^*}(t)$ is a circle with total length equal to the total angle at $\tilde{\gamma}_{v^*}(t)$. V_2 is between V_1 and W^- in the sense that it lies within the angle-$\frac{\pi}{2}$ arc of directions connecting V_1 and W^- in the space of directions. Thus, the angle between V_2 and W^- is less than or equal to $\frac{\pi}{2}$ and so the angle between V_2 and W^+ is at least $\frac{\pi}{2}$.
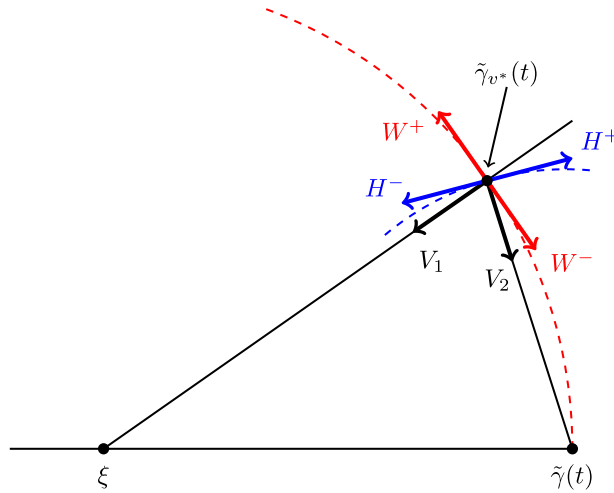
Fig. 4. The proof that $d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\gamma}_v(t))$ is non-decreasing in $|v|$.

If the angle between V_2 and W^+ is $\frac{\pi}{2}$, then the geodesic segment $[\tilde{\gamma}(t), \tilde{\gamma}_{v^*}(t)]$ must run through ξ and then for $v > v^*$, $d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\gamma}_v(t)) = 2(t - t_0) = d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\gamma}_{v^*}(t))$. If the angle is strictly less than $\frac{\pi}{2}$, then in the space of directions, W^+ is separated from V_2 by $H^{\pm}$. This means that as the path $v \mapsto \tilde{\gamma}_v(t)$ leaves the point $\tilde{\gamma}_{v^*}(t)$ with v increasing, it must move—at least initially—to the outside of the circle of radius $d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\gamma}_{v^*}(t))$ centered at $\tilde{\gamma}(t)$. In particular, $d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\gamma}_{v^*}(t))$ is locally monotonically increasing near v^*. As v^* was arbitrary (among v such that $\tilde{\gamma}_v$ give geodesic extensions of $\tilde{\gamma}(-\infty, t_0)$), and the path $v \mapsto \tilde{\gamma}_v(t)$ is connected, this completes our proof of the claim and the lemma. ∎

The 1st step in the dynamical argument for a pressure gap is the following technical proposition, which allows us to find a regular geodesic that is close to any connected component of the δ-neighborhood of the singular set.

Proposition 7.4. Let $\delta > 0$ and $0 < \eta < \frac{\eta_0}{2s}$ be given, where η_0 is defined in Lemma 2.15(b). Then there exists $L > 0$ and a family of maps $\Pi_t : \mathrm{Sing} \to \mathrm{Reg}(\eta)$ such that for all $t > 3L$ and for all $\gamma \in \mathrm{Sing}$, if we write $c = \Pi_t(\gamma)$ then the following are true:

  (a) $c, g_{t+t'}c \in \mathrm{Reg}(\eta)$ for some $|t'| < 4d_0$;

  (b) $d_{GS}(g_r c, \mathrm{Sing}) < \delta$ for all $r \in [L, t - L]$;

  (c) for all $r \in [L, t - L]$, $g_r c$ and γ lie in the same connected component of $B(\mathrm{Sing}, \delta)$, the δ-neighborhood of Sing.

Furthermore, $c(0), c(t + t') \in Con$, any $c \in \Pi_t(\text{Sing})$ is entirely determined (among the geodesics in $\Pi_t(\text{Sing})$) by the segment $c[0, t+t']$, and $d_{\tilde{S}}(\gamma(0), c(0)), d_{\tilde{S}}(\gamma(t), c(t+t')) < 2d_0$ where d_0 is as in Lemma 2.15(a).

Remark. The above proposition should be compared with [4, Theorem 8.1], although we have made two slight adjustments for our situation. First, we cannot guarantee that $g_t c \in \text{Reg}(\eta)$, but only that $g_{t+t'} c \in \text{Reg}(\eta)$ with uniform control on $|t'|$. Second, we prove our result for all $t > 3L$, instead of $2L$. These result in trivial changes to subsequent estimates in [4]'s argument.

Proof of Proposition 7.4. We begin with a geometric preliminary.

(A) Suppose that $\tilde{c}_1$ and $\tilde{c}_2$ are geodesic rays in $\tilde{S}$ with $\tilde{c}_1(0) = \tilde{c}_2(0)$ and $d_{\tilde{S}}(\tilde{c}_1(l), \tilde{c}_2(l)) \leq 3d_0$. The distance between geodesic rays is a convex function in a CAT(0) space, so $d_{\tilde{S}}(\tilde{c}_1(r), \tilde{c}_2(r)) \leq \frac{3d_0}{l} r$ for all $r \in [0, l]$. Therefore, to ensure that $d_{\tilde{S}}(\tilde{c}_1(r), \tilde{c}_2(r)) < \frac{\delta}{2}$ for all $r \in [0, 2T]$, it is sufficient to have $\frac{3d_0}{l} 2T < \frac{\delta}{2}$, or $l > \frac{12d_0 T}{\delta}$.

We now begin the proof in earnest. Let $\delta > 0$ and $0 < \eta < \frac{\eta_0}{2s}$ be given. Let $T(\delta)$ be as in Lemma 2.12. Let

$$L = \max\left\{ d_0, \frac{8d_0}{\eta_0}, 2T(\delta), \frac{12d_0 T(\delta)}{\delta} \right\};$$

we will highlight the need for each condition on L as we come to it in the proof. Let $t > 3L$, and let $\gamma \in \text{Sing}$.

As usual, we work in $\tilde{S}$. Let R be the maximal, isometrically embedded Euclidean rectangle with $\tilde{\gamma}([L, t - L])$ as one side, containing no cone points in its interior, and to the right side of $\tilde{\gamma}$, with respect to its orientation. (Throughout this proof, refer to Figure 5. For ease of exposition, we will often refer to the orientation as depicted in that figure in this proof.) Note that if $\tilde{\gamma}([L, t - L])$ contains any cone points with an angle $> \pi$ on the right side of $\tilde{\gamma}$, then R has height zero. That $t > 3L$ implies R has positive width. By maximality of R, there must be cone points on the boundary of R, specifically on the bottom side of R, as oriented in Figure 5. Let ξ_1 be the cone point closest to $\tilde{\gamma}(0)$ and ξ_2 be the cone point closest to $\tilde{\gamma}(t)$ on the bottom side of R.

Using Lemma 7.2, extend the bottom side of R to a singular geodesic $\tilde{\gamma}'$ that turns with angle π on the $\tilde{\gamma}$ side any time it encounters a cone point (i.e., measured from within the connected component of $\tilde{S} \setminus \tilde{\gamma}'$ containing $\tilde{\gamma}$, the incoming and outgoing directions
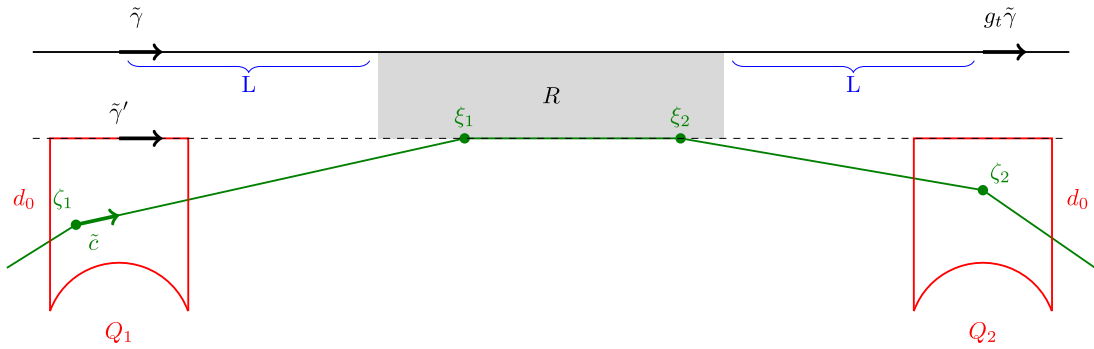
Fig. 5. The construction of $c = \Pi_t(\gamma)$ in Proposition 7.4.

of $\tilde{\gamma}'$ make angle π at any cone point). If R has height zero, let $\tilde{\gamma}' = \tilde{\gamma}$. Parametrize $\tilde{\gamma}'$ so that $\tilde{\gamma}'(L)$ is the lower-left corner of R (and hence $\tilde{\gamma}'(t-L)$ is the lower-right corner).

Construct geodesic segments of length d_0, starting at the points $\tilde{\gamma}'(-\frac{d_0}{2})$ and $\tilde{\gamma}'(\frac{d_0}{2})$, ending below $\tilde{\gamma}'$, and perpendicular to $\tilde{\gamma}'$ in the sense that for each segment, both angles between it and $\tilde{\gamma}'$ are $\geq \frac{\pi}{2}$. Connect the endpoints of these segments with a geodesic segment, forming a quadrilateral that we denote by Q_1. Construct a similar quadrilateral Q_2 based on $\tilde{\gamma}'$ around $\tilde{\gamma}'(t)$ on the same side as Q_1. Any point in Q_1 (resp. Q_2) can be reached from $\tilde{\gamma}'(0)$ (resp. $\tilde{\gamma}'(t)$) via a path along $\tilde{\gamma}'$ of length $\leq \frac{d_0}{2}$ followed by a perpendicular segment of length $\leq d_0$. Therefore, for any $\zeta \in Q_1$, $d_{\tilde{S}}(\tilde{\gamma}'(0), \zeta) \leq \frac{3}{2}d_0 < 2d_0$ (the analogous bound holds for Q_2) and the diameter of Q_i is bounded by $3d_0$. Our choice of $L \geq d_0$ implies that $\tilde{\gamma}'(L)$ and $\tilde{\gamma}'(t-L)$ are not in the quadrilaterals.

By their construction using d_0 from Lemma 2.15, the quadrilaterals Q_1 and Q_2 must contain cone points. Let ζ_1 be a cone point in Q_1 and ζ_2 a cone point in Q_2. Let $\hat{t} = d_{\tilde{S}}(\zeta_1, \zeta_2)$. Extend the geodesic segment $[\zeta_1, \zeta_2]$ to a geodesic $\tilde{c}$, parameterized so that $\tilde{c}(0) = \zeta_1$ and $\tilde{c}(\hat{t}) = \zeta_2$, with turning angles equal to exactly half of the total angle at each cone point ζ_1, ζ_2, and any cone points encountered over times $(-\infty, 0] \cup [\hat{t}, \infty)$. Note that this condition implies that c is determined entirely by the segment $[\zeta_1, \zeta_2]$. Then, $\tilde{c} \in \text{Reg}(\eta)$. An alternate path from ζ_1 to ζ_2 is to travel $\zeta_1 \to \tilde{\gamma}'(0) \to \tilde{\gamma}'(t) \to \zeta_2$ that has length $< 4d_0 + t$. Thus, $\hat{t} < t + 4d_0$. Reversing the roles of $\tilde{c}$ and $\tilde{\gamma}'$ also shows $t < \hat{t} + 4d_0$, so $\hat{t} = t + t'$ with $|t'| < 4d_0$. Then, $g_{t+t'}\tilde{c} \in \text{Reg}(\eta)$ as desired.

We claim that $[\xi_1, \xi_2] \subseteq \tilde{c} \cap R$. Consider the geodesic segments $[\zeta_1, \xi_1]$, $[\xi_1, \xi_2]$, and $[\xi_2, \zeta_2]$. The triangle formed by $\tilde{\gamma}'(0)$, ζ_1 and ξ_1 has $d_{\tilde{S}}(\tilde{\gamma}'(0), \xi_1) \geq L$ and as noted above, $d_{\tilde{S}}(\tilde{\gamma}'(0), \zeta_1) < 2d_0$. Using the CAT(0) property and an easy Euclidean geometry calculation, the angle between $[\tilde{\gamma}'(0), \xi_1]$ and $[\zeta_1, \xi_1]$ at ξ_1 is less than $\frac{4d_0}{L}$. Our assumption that $L \geq \frac{8d_0}{\eta_0}$ ensures that this angle is less than $\frac{\eta_0}{2}$. An analogous argument bounds the

angle between $[\xi_2, \tilde{\gamma}'(t)]$ and $[\xi_2, \zeta_2]$. By Lemma 2.15, there is excess angle at least η_0 at ξ_1 and ξ_2. At ξ_1 (and similarly at ξ_2, even if $\xi_1 = \xi_2$) the angle our concatenation of segments makes on the side toward $\tilde{\gamma}$ is at least the angle $\tilde{\gamma}'$ makes on that side, which by construction is π. On the side away from $\tilde{\gamma}$, the angle our concatenation makes is at least $\mathcal{L}(\xi_1) - \pi - \eta_0 > \pi$. The concatenation of $[\zeta_1, \xi_1]$, $[\xi_1, \xi_2]$, and $[\xi_2, \zeta_2]$ is therefore a geodesic segment, and hence it must be a subsegment of $\tilde{c}$, proving the claim.

We now need to show, using our choice of L, that $d_{G\tilde{S}}(g_{r^*}\tilde{c}, \text{Sing}) < \delta$ for all $r^* \in [L, t - L]$. We do this by showing that for each such r^*, there is a geodesic $\tilde{\gamma}' \in \text{Sing}$ such that $d_{\tilde{S}}(\tilde{\gamma}'(r), \tilde{c}(r)) < \frac{\delta}{2}$ for all $r \in [r^* - T(\delta), r^* + T(\delta)]$ and then invoking Lemma 2.12.

Let $\tilde{\gamma}_0'$ be the reparameterization of $\tilde{\gamma}'$ so that $\tilde{c}(r) = \tilde{\gamma}_0'(r)$ whenever $\tilde{c}(r) \in R$. Let $[r_1, r_2] = \{r : \tilde{c}(r) = \tilde{\gamma}_0'(r) \in \tilde{c} \cap R\}$. (Figure 5 depicts a situation where $\tilde{c}(r_1) = \xi_1$ and $\tilde{c}(r_2) = \xi_2$.) For any $r \in [r_1, r_2]$, consider the geodesic rays $\tilde{c}(-\infty, r)$ and $\tilde{\gamma}_0'(-\infty, r)$. They share the point $\tilde{c}(r) = \tilde{\gamma}_0'(r)$ and at some distance $\geq L \geq \frac{12 d_0 T(\delta)}{\delta}$ are both in Q_1 and hence $\leq 3d_0$ apart (with respect to $d_{\tilde{S}}$). Therefore, by (A) at the start of this proof, for all $r \in [r_1 - 2T(\delta), r_2]$, $d_{\tilde{S}}(\tilde{c}(r), \tilde{\gamma}_0'(r)) < \frac{\delta}{2}$. Applying the same argument to the rays $\tilde{c}(r, \infty)$ and $\tilde{\gamma}_0'(r, \infty)$, shows $d_{\tilde{S}}(\tilde{c}(r), \tilde{\gamma}_0'(r)) < \frac{\delta}{2}$ for all $r \in [r_1, r_2 + 2T(\delta)]$. As $\tilde{\gamma}_0' \in \text{Sing}$, by Lemma 2.12, $d_{G\tilde{S}}(g_{r^*}\tilde{c}, \text{Sing}) < \delta$ for all $r^* \in [r_1 - T(\delta), r_2 + T(\delta)]$.

If this covers all times in $[L, t - L]$, we are done with this part of the proof. If not, we continue as follows. Assuming $r_1 - T(\delta) > L$, consider the geodesic segment $[\zeta_1, \xi_1]$. Let $[\xi_1^-, \xi_1]$ be the maximal subsegment of $[\zeta_1, \xi_1]$ containing no cone points in its interior. Extend $[\xi_1^-, \xi_1]$ to a geodesic $\tilde{\gamma}_{-1}'$ in Sing lying between $\tilde{c}$ and $\tilde{\gamma}_0'$, parametrized so that $\tilde{\gamma}_{-1}'(r_1) = \xi_1 = \tilde{c}(r_1)$. First, note that over the interval $[r_1 - 2T(\delta), r_1]$, $\tilde{\gamma}_{-1}'(r)$ is at least as close to $\tilde{\gamma}_0'(r)$ as $\tilde{c}(r)$ is, and by our work above, this distance is bounded above by $\frac{\delta}{2}$. By Lemma 2.12, $d_{G\tilde{S}}(g_{r_1 - T(\delta)}\tilde{\gamma}_0', g_{r_1 - T(\delta)}\tilde{\gamma}_{-1}') < \delta$. Second, we can argue regarding $\tilde{c}$ and $\tilde{\gamma}_{-1}'$ exactly as we did regarding $\tilde{c}$ and $\tilde{\gamma}_0'$. They form rays with a common endpoint which after some distance $> L$ are still within $3d_0$ of each other, which as noted above allows us to show they are δ close in $d_{G\tilde{S}}$ for an interval of time below r_1. This interval will either extend to L as desired or will end at some $r_0 - T(\delta)$ where $\tilde{c}$ and $\tilde{\gamma}_{-1}'$ branch apart at a cone point. We then repeat our argument at that cone point, finding $\tilde{\gamma}_{-2}' \in \text{Sing}$ shadowing $\tilde{c}$ further, and so on, until we have reached time L. Exactly the same argument applies beyond ξ_2, constructing $\tilde{\gamma}_1', \tilde{\gamma}_2', \ldots \in \text{Sing}$ as necessary to shadow $\tilde{c}$ in $d_{G\tilde{S}}$ until time $t - L$.

It remains to establish that for all $r \in [L, t - L]$, $g_r\tilde{c}$ and $\tilde{\gamma}$ lie in the same connected component of $B(\text{Sing}, \delta)$. We do this by showing that one can get from $\tilde{\gamma}$ to $g_r\tilde{c}$ by a series of "moves", each of which can be realized by a continuous path in $B(\text{Sing}, \delta)$.

Move 1: geodesic flow

If $\tilde{\gamma} \in \mathrm{Sing}$, then for all r, $g_r\tilde{\gamma} \in \mathrm{Sing}$ with the flow itself providing a continuous path between the two, so $\tilde{\gamma}$ and $g_r\tilde{\gamma}$ are both in the same connected component of Sing itself and hence of $B(\mathrm{Sing}, \delta)$.

Move 2: "pivot"

Suppose $\tilde{\gamma}_i, \tilde{\gamma}_{i+1} \in \mathrm{Sing}$ with $\tilde{\gamma}_i(0) = \tilde{\gamma}_{i+1}(0) = \xi \in \widetilde{Con}$, $d_{G\tilde{S}}(\tilde{\gamma}_i, \tilde{\gamma}_{i+1}) < \delta$, and suppose that the angle between $\tilde{\gamma}_i$ and $\tilde{\gamma}_{i+1}$ at ξ is less than $\mathcal{L}(\xi) - 2\pi$. Note that any geodesic ray starting from ξ that lies between $\tilde{\gamma}_i(-\infty, 0)$ and $\tilde{\gamma}_{i+1}(-\infty, 0)$ can be concatenated with $\tilde{\gamma}_i(0, +\infty)$ to form a geodesic. Similarly, any ray between $\tilde{\gamma}_i(0, +\infty)$ and $\tilde{\gamma}_{i+1}(0, +\infty)$ can be concatenated with $\tilde{\gamma}_{i+1}(-\infty, 0)$ to form a geodesic.

Let $\tilde{\gamma}_{i+1} \cdot \tilde{\gamma}_i$ be the concatenation of $\tilde{\gamma}_{i+1}(-\infty, 0]$ with $\tilde{\gamma}_i[0, +\infty)$. Note that $d_{G\tilde{S}}(\tilde{\gamma}_i, \tilde{\gamma}_{i+1} \cdot \tilde{\gamma}_i)$ and $d_{G\tilde{S}}(\tilde{\gamma}_{i+1} \cdot \tilde{\gamma}_i, \tilde{\gamma}_{i+1})$ are both less than $d_{G\tilde{S}}(\tilde{\gamma}_i, \tilde{\gamma}_{i+1})$ and hence less than δ. Indeed, the integrals computing $d_{G\tilde{S}}(\tilde{\gamma}_i, \tilde{\gamma}_{i+1} \cdot \tilde{\gamma}_i)$ and $d_{G\tilde{S}}(\tilde{\gamma}_{i+1} \cdot \tilde{\gamma}_i, \tilde{\gamma}_{i+1})$ will each match the integral to compute $d_{G\tilde{S}}(\tilde{\gamma}_i, \tilde{\gamma}_{i+1})$ on one side of $t = 0$, and will replace the integral on the other side of $t = 0$ by zero, if anything decreasing the distance.

We "pivot" from $\tilde{\gamma}_i$ to $\tilde{\gamma}_{i+1}$ in two steps. First, let ζ_v be a continuous and monotonic path in $\partial_\infty \tilde{S}$ from $\zeta_0 = \tilde{\gamma}_i(-\infty)$ to $\zeta_1 = \tilde{\gamma}_{i+1}(-\infty)$. Apply Lemma 7.3 to get a continuous path $v \mapsto \tilde{c}_v$ from $\tilde{\gamma}_i$ to $\tilde{\gamma}_{i+1} \cdot \tilde{\gamma}_i$ such that for all v, $d_{G\tilde{S}}(\tilde{\gamma}_i, \tilde{c}_v) \leq d_{G\tilde{S}}(\tilde{\gamma}_i, \tilde{\gamma}_{i+1} \cdot \tilde{\gamma}_i) < \delta$. Second, let ζ'_v be a continuous and monotonic path from $\tilde{\gamma}_i(+\infty)$ to $\tilde{\gamma}_{i+1}(+\infty)$, and apply Lemma 7.3 to get a continuous path $v \mapsto \tilde{c}'_v$ from $\tilde{\gamma}_{i+1} \cdot \tilde{\gamma}_i$ to $\tilde{\gamma}_{i+1}$. Again, for all v, $d_{G\tilde{S}}(\tilde{c}'_v, \tilde{\gamma}_{i+1}) < d_{G\tilde{S}}(\tilde{\gamma}_{i+1} \cdot \tilde{\gamma}_i, \tilde{\gamma}_{i+1}) < \delta$; this time, we apply the distance nonincreasing property obtained in Lemma 7.3 to the reverse of the path $v \mapsto \tilde{c}'_v$, which continuously moves from $\tilde{\gamma}_{i+1}$ to $\tilde{\gamma}_{i+1} \cdot \tilde{\gamma}_i$. Overall, we have a path of geodesics that remains in $B(\mathrm{Sing}, \delta)$ throughout.

Move 3: "slide"

Suppose that R is an isometrically embedded Euclidean rectangle in $\tilde{S}$. (Note that this implies R contains no cone points in its interior.) Let $\tilde{\gamma}, \tilde{\gamma}' \in \mathrm{Sing}$ be geodesics that extend the top and bottom sides of R, respectively, with $d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\gamma}') < \delta$. Let $\{c_v\}$ be a continuous path of horizontal (i.e., parallel to $\tilde{\gamma}$ and $\tilde{\gamma}'$ within R) geodesic segments connecting the two sides of R, which move monotonically downward through R, with $c_0 = \tilde{\gamma} \cap R$ and $c_1 = \tilde{\gamma}' \cap R$.

For each v, let $\tilde{\gamma}_v^u$ be the "uppermost" geodesic extension of c_v, that is, the extension which turns with angle π on the $\tilde{\gamma}$-side at any cone point it hits. Let $\tilde{\gamma}_v^l$ be the "lowermost" geodesic extension of c_v, that is, the extension that turns with angle π

on the $\tilde{\gamma}'$-side at any cone point it hits. Since the distance between geodesics is a convex function and since c_v is parallel to $\tilde{\gamma}$ and $\tilde{\gamma}'$ over R, both $\tilde{\gamma}_v^u$ and $\tilde{\gamma}_v^l$ lie between $\tilde{\gamma}$ and $\tilde{\gamma}'$.

If $\tilde{\gamma}_v^u = \tilde{\gamma}_v^l$, set $\tilde{\gamma}_v = \tilde{\gamma}_v^u = \tilde{\gamma}_v^l$. This happens if and only if $\tilde{\gamma}_v$ hits no cone points. Since there are countably many cone points in $\tilde{S}$, there is a countable set $\{v_n\} \subset [0,1]$ for which $\tilde{\gamma}_{v_n}^u \neq \tilde{\gamma}_{v_n}^l$. Let $\{I_n\}$ be a corresponding collection of closed real intervals with $\sum |I_n| = 1$. Cut $[0,1]$ at each v_n and glue in the interval I_n, resulting in an interval of length 2. Adjust the subscripts where $\tilde{\gamma}_v$ has already been defined accordingly. For each n, if $I_n = [a_n, b_n]$ set $\tilde{\gamma}_{a_n} = \tilde{\gamma}_{v_n}^u$ and $\tilde{\gamma}_{b_n} = \tilde{\gamma}_{v_n}^l$. For all $v \in I_n$, use Lemma 7.3 to fill in a continuous path $v \mapsto \tilde{\gamma}_v$ from $\tilde{\gamma}_{a_n}$ to $\tilde{\gamma}_{b_n}$.

The result is a path $v \mapsto \tilde{\gamma}_v$ from $\tilde{\gamma}$ to $\tilde{\gamma}'$ that we claim is continuous. Continuity at any v_0 that is in the interior of one of the inserted intervals I_n is provided by Lemma 7.3. If v_0 is on the boundary of some I_n and v approaches v_0 from inside I_n, Lemma 7.3 again applies. Otherwise, $\tilde{\gamma}_{v_0}$ is in Sing and as $v \to v_0$, $\tilde{\gamma}_v$ approaches $\tilde{\gamma}_{v_0}$ from a side on which $\tilde{\gamma}_{v_0}$ always turns with angle π. In this case, let $\varepsilon > 0$ be given. Since there are only finitely many cone points in any compact region of $\tilde{S}$, for v sufficiently close to v_0, there are no cone points in the convex hull of $\tilde{\gamma}_{v_0}[-T(\varepsilon), T(\varepsilon)]$ and $\tilde{\gamma}_v[-T(\varepsilon), T(\varepsilon)]$. Perhaps, making v even closer to v_0, this convex hull is a rectangle with width $< \frac{\varepsilon}{2}$. Then, by Lemma 2.12, $d_{G\tilde{S}}(\tilde{\gamma}_v, \tilde{\gamma}_{v_0}) < \varepsilon$, proving continuity at v_0.

Finally, we claim that $d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\gamma}_v)$ is nondecreasing. Let $a < b$ be in $[0,2]$. If $a, b \in I_n$, $d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\gamma}_a) \leq d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\gamma}_b)$ by Lemma 7.3. Therefore, to prove the distance is nondecreasing in general, we just need to show $d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\gamma}_a) \leq d_{G\tilde{S}}(\tilde{\gamma}, \tilde{\gamma}_b)$ when a and b are close and a is the lower endpoint of some I_n or is in the complement of the $\{I_n\}$. In either case, $\tilde{\gamma}_a$ is a singular geodesic that makes angle π at any cone points it encounters on the side away from $\tilde{\gamma}$. For each fixed t, consider the geodesic segment $c_{v,t} = [\tilde{\gamma}(t), \tilde{\gamma}_v(t)]$ and how it varies with v. We claim the length of $c_{v,a}$ is at most the length of $c_{v,t}$ for small enough $t > a$, which together with the formula for $d_{G\tilde{S}}$ will establish the desired result. As v increases from a, $\tilde{\gamma}_v(t)$ will move along a geodesic path perpendicular to $\tilde{\gamma}_a$ on the side of $\tilde{\gamma}_a$ away from $\tilde{\gamma}$. Indeed, for all $b > a$ small enough that no cone points are in the convex hull of $\tilde{\gamma}_a[0,t]$ and $\tilde{\gamma}_b[0,t]$, by construction, $\tilde{\gamma}_b[0,t]$ will simply be the translation of $\tilde{\gamma}_a[0,t]$ across an embedded Euclidean rectangle. Take such a $b > a$. Then, consider the geodesic triangle with sides $c_{a,t}$, $c_{b,t}$, and $[\tilde{\gamma}_a(t), \tilde{\gamma}_b(t)]$. Since $[\tilde{\gamma}_a(t), \tilde{\gamma}_b(t)]$ is perpendicular to $\tilde{\gamma}_a$ on the side away from $\tilde{\gamma}$ and $c_{a,t}$ hits $\tilde{\gamma}_a(t)$ from the side toward $\tilde{\gamma}$, the angle between $c_{a,t}$ and $[\tilde{\gamma}_a(t), \tilde{\gamma}_b(t)]$ is at least $\frac{\pi}{2}$. By comparison with a Euclidean triangle and the CAT(0) property, $c_{b,t}$ is longer than $c_{a,t}$ giving the desired result.

Therefore, $\tilde{\gamma}_v$ is in the same connected component of $B(\text{Sing}, \delta)$ for all v for this "slide" move.

We return now to our construction of $\tilde{c}$. For any $r \in [L, t-L]$, we can reach g_r via the following series of the moves noted above. First, we move $\tilde{\gamma} \to g_{t/2}\tilde{\gamma}$ by geodesic flow. Second, we slide $g_{t/2}\tilde{\gamma}$ down across R (if R has nonzero height) to a geodesic in the orbit of $\tilde{\gamma}_0'$ using our "slide" move. We break this move down into a sequence of small "slide" moves between geodesics $\tilde{\gamma}_{v_n}$ in Sing. Since $t > 3L$ and $L \geq 2T(\delta)$ if we choose v_n so that $\tilde{\gamma}_{v_n} \cap R$ and $\tilde{\gamma}_{v_{n+1}} \cap R$ are within $\delta/2$ vertically in R, by Lemma 2.12, $d_{G\tilde{S}}(\tilde{\gamma}_{v_n}, \tilde{\gamma}_{v_{n+1}}) < \delta$. Therefore, this series of moves stays in the same connected component of $B(\text{Sing}, \delta)$. Finally, we apply a series of "pivot" moves and the geodesic flow to get to $g_r c$ via the geodesics $\tilde{\gamma}_i'$ introduced in our construction above. Our work in the construction showed that all the "pivot" moves involved are between geodesics within δ of one another. Therefore, in total, we have a continuous path from $\tilde{\gamma}$ to $g_r \tilde{c}$ in $B(\text{Sing}, \delta)$, completing the proof of Proposition 7.4. ∎

The 2nd step in the argument for the pressure gap is to prove the following Lemma, which uniformly controls how many geodesics in Sing can have image under Π_t near to a fixed geodesic. Recall that

$$d_{GS,t}(\gamma_1, \gamma_2) = \max_{s \in [0,t]} d_{GS}(g_s \gamma_1, g_s \gamma_2)$$

and that a subset of GS is $(t, 2\varepsilon)$-separated if its members are pairwise distance at least 2ε apart with respect to $d_{GS,t}$.

Lemma 7.5 (Compare with Prop. 8.2 in [4]). For all $\varepsilon > 0$, there exists some $C(\varepsilon) > 0$ such that if $E_t \subset \text{Sing}$ is a $(t, 2\varepsilon)$-separated set for some $t > 3L$, then for any $w \in GS$,

$$\#\{\gamma \in E_t \mid d_{GS,t}(w, \Pi_t(\gamma)) < \varepsilon\} \leq C.$$

Proof.   It is sufficient to prove the result in $G\tilde{S}$.

Let d_0 be as in Lemma 2.15(a). Fix $w \in G\tilde{S}$, and let $\varepsilon > 0$, $t > 3L$, and E_t be given.

Suppose that $d_{GS,t}(w, c) < \varepsilon$. Then, by definition, $d_{G\tilde{S}}(g_r w, g_r c) < \varepsilon$ for all $r \in [0, t]$. By Lemma 2.11, that $d_{G\tilde{S}}(w, c) < \varepsilon$ implies $d_{\tilde{S}}(w(0), c(0)) < 2\varepsilon$ and that $d_{G\tilde{S}}(g_t w, g_t c) < \varepsilon$ implies $d_{\tilde{S}}(w(t), c(t)) < 2\varepsilon$.

By Proposition 7.4, any geodesic c in $\Pi_t(\text{Sing})$ has $c(0) \in \widetilde{Con}$ and $c(t + t') \in \widetilde{Con}$ for some $|t'| < 4d_0$. Using what we noted above, the cone point at $c(0)$ must be within $d_{\tilde{S}}$-distance 2ε of $w(0)$ and the cone point at $c(t + t')$ must be within $d_{\tilde{S}}$-distance $2\varepsilon + 4d_0$ of $w(t)$.

As S is compact and *Con* is a discrete subset, for any $R > 0$, $N_R = \max_{p \in \tilde{S}} \#\{\widetilde{Con} \cap B_R(p)\}$ is finite. Let $C_1(\varepsilon) = N_{2\varepsilon} N_{2\varepsilon + 4d_0}$. As specified in Proposition 7.4, any element c of $\Pi_t(\text{Sing})$ is entirely determined by the cone points $c(0)$ and $c(t + t')$. Thus, there are at most $C_1(\varepsilon)$ elements $c \in \Pi_t(\text{Sing})$ with $d_{G\tilde{S}}(w, c) < \varepsilon$.

Now we want to bound $\#\{\gamma \in E_t \mid \Pi_t(\gamma) = c\}$ for any $c \in \Pi_t(\text{Sing})$. For $\gamma \in E_t$, the construction of $\Pi_t(\gamma)$ shows that $d_{\tilde{S}}(\gamma(0), c(0)) < 2d_0$ and $d_{\tilde{S}}(\gamma(t), c(t + t')) < 2d_0$. Therefore, $\gamma(-T(\varepsilon)) \in B(c(0), 2d_0 + T(\varepsilon))$ and $\gamma(t + T(\varepsilon)) \in B(c(t + t'), 2d_0 + T(\varepsilon))$, where $T(\varepsilon)$ is as in Lemma 2.12. Let P be an $\frac{\varepsilon}{8}$-spanning set for $B(c(0), 2d_0 + T(\varepsilon))$ with respect to $d_{\tilde{S}}$ and Q an $\frac{\varepsilon}{8}$-spanning set for $B(c(t + t'), 2d_0 + T(\varepsilon))$ with respect to $d_{\tilde{S}}$. By the compactness of S, there exists some $C_2(\varepsilon)$ such that $\#P$ and $\#Q$ are bounded above by $C_2(\varepsilon)$. For each $(p, q) \in P \times Q$, extend $[p, q]$ to a geodesic $\eta_{p,q}$ with $\eta_{p,q}(-T(\varepsilon)) = p$.

Since P and Q are $\frac{\varepsilon}{8}$-spanning, there exist $(p, q) \in P \times Q$ such that $d_{\tilde{S}}(\gamma(-T(\varepsilon)), p) < \frac{\varepsilon}{8}$ and $d_{\tilde{S}}(\gamma(t + T(\varepsilon)), q) < \frac{\varepsilon}{8}$. We immediately have that $d_{\tilde{S}}(\gamma(-T(\varepsilon)), \eta_{p,q}(-T(\varepsilon))) < \frac{\varepsilon}{8}$. In addition, $\gamma[-T(\varepsilon), t + T(\varepsilon)]$ and $[p, q]$ are geodesic segments whose endpoints are each less than $\frac{\varepsilon}{8}$ apart. Since geodesic segments in $\tilde{S}$ minimize length, the length of $[p, q]$ is within $\frac{\varepsilon}{4}$ of $t + 2T(\varepsilon)$, the length of $\gamma[-T(\varepsilon), t + T(\varepsilon)]$. Therefore, we also have

$$d_{\tilde{S}}(\gamma(t + T(\varepsilon)), \eta_{p,q}(t + T(\varepsilon))) \leq d_{\tilde{S}}(\gamma(t + T(\varepsilon)), q) + d_{\tilde{S}}(q, \eta_{p,q}(t + T(\varepsilon))$$
$$< \frac{\varepsilon}{8} + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}.$$

Using convexity of the distance between geodesics in a CAT(0) space and our bounds on the distances between the pairs of endpoints, we have

$$d_{\tilde{S}}(\gamma(r), \eta_{p,q}(r)) < \frac{\varepsilon}{2} \quad \text{for all} \quad r \in [-T(\varepsilon), t + T(\varepsilon)].$$

Then, by Lemma 2.12, $d_{G\tilde{S}}(g_r\gamma, g_r\eta_{p,q}) < \varepsilon$ for all $r \in [0, t]$, or, equivalently, $d_{G\tilde{S},t}(\gamma, \eta_{p,q}) < \varepsilon$.

We can conclude that $\#\{\gamma \in E_t \mid \Pi_t(\gamma) = c\} \leq \#\{\eta_{p,q}\} \leq C_2(\varepsilon)^2$. Indeed, if there are more than $C_s(\varepsilon)^2$ elements in E_t that have image c under Π_t, then some two of them must both be within $d_{G\tilde{S},t}$-distance ε of the same $\eta_{p,q}$ and hence less than 2ε apart with respect to $d_{G\tilde{S},t}$, contradicting the fact that E_t is $(t, 2\varepsilon)$-separated.

Putting these estimates together, $\#\{\gamma \in E_t \mid d_{GS,t}(w, \Pi_t(\gamma)) < \varepsilon\} \leq C_1(\varepsilon)C_2(\varepsilon)^2$, completing the proof. ∎

The 3rd step of the argument closely follows [4], as we now outline. First, by Lemmas 4.1 and 4.2 of [10], for any $\varepsilon > 0$ and $t > 0$,

$$\sup \left\{ \sum_{\gamma \in E} e^{\sup_{\xi \in B_t(\gamma, \varepsilon)} \int_0^t \phi(g_r \xi) dr} \,\middle|\, E \subset \text{Sing is } (t, \varepsilon)\text{-separated} \right\} \geq e^{tP(\text{Sing}, 2\varepsilon, \phi)}. \quad (12)$$

To apply this fact from [10] here, we just need to recall that Sing is compact (noted in Definition 2.4).

We now use the fact that ϕ is locally constant on a neighborhood of Sing. For sufficiently small ε, the left-hand side of the inequality above is equal to

$$\Lambda(\text{Sing}, \phi, \varepsilon, t) := \sup \left\{ \sum_{\gamma \in E} e^{\int_0^t \phi(g_r \gamma) dr} \,\middle|\, E \subset \text{Sing is } (t, \varepsilon)\text{-separated} \right\}. \quad (13)$$

Combining (12) and (13) and using the fact that g_t is entropy-expansive (Lemma 2.17) exactly as in [4], for sufficiently small ε,

$$\Lambda(\text{Sing}, \phi, \varepsilon, t) \geq e^{tP(\text{Sing}, \phi)}. \quad (14)$$

Fix $0 < \eta < \frac{\eta_0}{2}$ where η_0 is from Lemma 2.15(b). Note that $\text{Reg}(\eta)$ has non-empty interior. Pick $\delta > 0$ small enough that $\Lambda(\text{Sing}, \phi, 2\delta, t) \geq e^{tP(\text{Sing}, \phi)}$, ϕ is locally constant on $B(\text{Sing}, \delta)$, and by Lemma 3.10, $\lambda(\gamma) < \eta$ for all $\gamma \in B(\text{Sing}, 2\delta)$. Then we proceed exactly as in [4], invoking Proposition 7.4 as a direct replacement of their Theorem 8.1 and Lemma 7.5 as a direct replacement for their Proposition 8.2. The argument produces the following lemma.

Lemma 7.6 (Lemma 8.4 in [4]). For sufficiently small $\delta > 0$, there is a $(t, 2\delta)$-separated set E_t in Sing such that there is a (t, δ)-separated set $E_t'' \subset \Pi_t(E_t)$ satisfying

$$\sum_{w \in E_t''} e^{\inf_{u \in B_t(w, \delta)} \int_0^t \phi(g_s u) ds} \geq \beta e^{tP(\text{Sing}, \phi)},$$

where $\beta = \frac{1}{C} e^{-6L\|\phi\|}$ and C is as in Lemma 7.5.

Proof. The only minor change needed in substituting our Proposition 7.4 for their Theorem 8.1 is to note that our condition on t is that it be $> 3L$, whereas theirs is that

it be $> 2L$. This gives us $\beta = \frac{1}{C}e^{-6L\|\phi\|}$ instead of $\beta = \frac{1}{C}e^{-4L\|\phi\|}$. This results in merely cosmetic changes to the rest of the argument in [4].   ∎

Note that $\{(w, t) : w \in E_t''\}$ is in $\mathcal{G}^{4d_0}(\eta)$, using the notation of Definition 5.7.

The final step in the argument is to use specification to string together orbit segments from E_t'' in many different orders so as to produce a large collection of long orbit segments that together produce more pressure than $P(\mathrm{Sing}, \phi)$. In [4], this is undertaken in Section 8.4, and at this point, the argument is almost entirely dynamical. It uses the estimate of Lemma 7.6 together with strong specification for $\mathcal{G}^{4d_0}(\eta)$ as given by Corollary 5.8. The one geometric piece of information used is that $\lambda(\gamma) < \eta$ for all $\gamma \in B(\mathrm{Sing}, 2\delta)$. Hence, we assumed this when choosing δ above, invoking Lemma 3.10. This completes the proof of Theorem 7.1.

Applying Theorem 7.1 with $\phi = 0$ gives the following.

Corollary 7.7. $h_{top}(g_t|_{\mathrm{Sing}}) < h_{top}(g_t)$.

With the pressure gap condition for such potentials in hand we briefly note a second class of potentials for which it holds. Proposition 4.7 of [6] notes that if the pressure gap $P(\mathrm{Sing}, \phi) < P(\phi)$ holds for ϕ, then for any function sufficiently close to ϕ (specifically with $2\|\phi - \psi\| < P(\phi) - P(\mathrm{Sing}, \phi)$) and any constant c, $P(\mathrm{Sing}, \psi + c) < P(\psi + c)$. Applying this to the locally constant functions ϕ discussed in this section gives us a further class of potentials with a pressure gap. Applying it with $\phi = 0$ gives us one class of particular note.

Corollary 7.8.   If ψ is a continuous potential with $\|\psi\| < \frac{1}{2}\left(h_{top}(g_t) - h_{top}(g_t|_{\mathrm{Sing}})\right)$, where h_{top} is the topological entropy, then $P(\mathrm{Sing}, \psi) < P(\psi)$.

8   Equilibrium States are Limits of Weighted Periodic Orbits

We can show that weighted periodic orbits equidistribute to the equilibrium states we have constructed, following a method of [4]. Throughout this section, we write $\mathcal{G}^M := \mathcal{G}^M(\eta)$ (see Definition 5.7) as we will work with a fixed η throughout.

Define the equivalence class of a closed geodesic $[\gamma]$ to be all geodesics $\eta \in GS$ for which $\gamma = g_t\eta$ for some $t \in \mathbb{R}$. Then let $\mathrm{Per}_R[Q - \delta, Q]$ be the set of equivalence classes of regular closed geodesics with period in $[Q - \delta, Q]$. Now consider such a regular closed geodesic, and define μ_γ to be the normalized Lebesgue measure supported on γ and $\Phi(\gamma) = \int_0^{\ell(\gamma)} \phi(g_u\gamma)\, du$. These definitions agree for all representatives of an equivalence

class, so we define $\mu_{[\gamma]} = \mu_\gamma$ and $\Phi([\gamma]) = \Phi(\gamma)$. We consider the weighted sum

$$\mu_{Q,\delta} = \frac{1}{\Lambda_R(Q,\delta,\phi)} \sum_{[\gamma] \in \mathrm{Per}_R[Q-\delta,Q]} e^{\Phi([\gamma])} \mu_{[\gamma]},$$

where $\Lambda_R(Q,\delta,\phi) = \sum\limits_{[\gamma] \in \mathrm{Per}_R[Q-\delta,Q]} e^{\Phi([\gamma])}$ is our normalizing constant. When $\lim\limits_{Q \to \infty} \frac{1}{Q} \log \Lambda_R(Q,$
$\delta, \phi)$ exists, it can be thought of as the pressure of closed saddle connection paths, and we write it as $P_{R,\delta}(\phi)$.

Theorem 8.1.   We use the notation above. Let ϕ be a Hölder potential with $P(\mathrm{Sing}, \phi) <$
$P(\phi)$, and let μ be the unique equilibrium state for ϕ. Then, for all $\delta > 0$, $P_{R,\delta}(\phi) = P(\phi)$
and in the weak-* topology we have $\lim\limits_{Q \to \infty} \mu_{Q,\delta} = \mu$.

Remark.   Note that this provides a way to identify interesting potentials, by considering geometrically relevant ways to weight closed geodesics. For instance, one could potentially try to identify a continuous function that weights γ by the number of conical points it turns at.

We first prove a lemma that will be necessary throughout this section.

Lemma 8.2.   Let 2ε be less than the injectivity radius of S. For all $Q \gg \delta > 0$, any set of representatives of the equivalence classes in $\mathrm{Per}_R[Q - \delta, Q]$ is (Q, ε)-separated.

Proof.   Consider $[\gamma_1], [\gamma_2] \in \mathrm{Per}_R[Q-\delta, Q]$, and let γ_1, γ_2 be representatives. Furthermore, suppose $d_{GS}(g_t \gamma_1, g_t \gamma_2) < \varepsilon$ for all $t \in [0, Q]$. By Lemma 2.11, $d_S(\gamma_1(t), \gamma_2(t)) < 2\varepsilon$ for all $t \in [0, Q]$. By our choice of ε, these geodesics are freely homotopic and represent the same element g of the fundamental group. Letting $\tilde{\gamma}_i$ be lifts of γ_i, we have that both $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are axes of g. By [3, Theorem II.6.8], $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are parallel, and so they bound a flat strip by the flat strip theorem. This contradicts the assumption that γ_1 and γ_2 are regular.   ∎

We have the following proposition, which follows from the proof of variational principle found in [21, Theorem 9.10] because $\mathrm{Per}_R[Q - \delta, Q]$ is (Q, ε)-separated for all sufficiently small ε.

Proposition 8.3.   If μ is the unique equilibrium state for ϕ, then for all $\delta > 0$ such that $\lim\limits_{Q \to \infty} \frac{1}{Q} \log \Lambda_R(Q, \delta, \phi) = P(\phi)$, we have $\lim\limits_{Q \to \infty} \mu_{Q,\delta} = \mu$.

In order to apply this proposition, we need to establish a growth rate for $\Lambda_R(Q, \delta, \phi)$ for all sufficiently small $\delta > 0$, which is done in Propositions 8.7 and 8.8 below.

First, we show that the growth rate for $\Lambda_R(Q, \delta, \phi)$ is fast enough. In order to do this, we need to be able to approximate $(\gamma, t) \in \mathcal{G}^M$ by closed geodesics of a bounded length. This is encapsulated in the following proposition.

Proposition 8.4.   For all $\delta > 0$, there exists T' such that for all $(\gamma, t) \in \mathcal{G}^M$ with $t > \frac{\theta_0}{\eta} + 2M$, there is some regular closed geodesic ξ with period in $[t + T' - \delta, t + T']$ such that $d_{GS}(g_u\gamma, g_u\xi) < \delta$ for all $u \in [0, t]$.

Proof.   First, we explain how to obtain the statement of the proposition for $(\gamma, t) \in \mathcal{G}$. Let $\delta > 0$, and let $\hat{\tau}$ be the specification constant for $\mathcal{G}$ with shadowing scale $\frac{\delta}{8}$ (see Proposition 5.6). We will show that $T' = \hat{\tau} + \frac{\delta}{2}$ satisfies the requirements of the Proposition. Let $(\gamma, t) \in \mathcal{G}$, and let ξ be a geodesic guaranteed by specification that shadows (γ, t) twice in succession. Now recall from Proposition 5.6 that there exists a closed interval $I \supset [\frac{\theta_0}{2\eta}, t - \frac{\theta_0}{2\eta}]$ such that ξ contains two copies of $\gamma(I)$. In other words, there exist $r_1, r_2 > 0$ such that $\xi(r_i + r) = \gamma(r)$ for all $r \in I$, where $i \in \{1, 2\}$. Thus, we can choose ξ to be a closed geodesic and observe that its length is given by $r_2 - r_1$. Now, since $d_{GS}(g_{\frac{\theta_0}{2\eta}}\xi, g_{\frac{\theta_0}{2\eta}}\gamma) \leq \frac{\delta}{8}$ by Proposition 5.6, we can apply Lemma 2.11 to show $d_S(\xi(\frac{\theta_0}{2\eta}), \gamma(\frac{\theta_0}{2\eta})) \leq \frac{\delta}{4}$. Thus, $|r_1| \leq \frac{\delta}{4}$. Similarly, considering the 2nd copy of (γ, t) that ξ shadows, we have $d_S(\xi(\frac{\theta_0}{2\eta} + t + \hat{\tau}), \gamma(\frac{\theta_0}{2\eta})) \leq \frac{\delta}{4}$, and so $r_2 \in [\hat{\tau} + t - \frac{\delta}{4}, \hat{\tau} + t + \frac{\delta}{4}]$. Hence, ξ is a regular closed geodesic with length in $[\hat{\tau} + t - \frac{\delta}{2}, \hat{\tau} + t + \frac{\delta}{2}]$. Taking $T' = \hat{\tau} + \frac{\delta}{2}$, we are done. In order to adapt this argument to $\mathcal{G}^M$ for $\tau > 0$, note that we achieve specification for $\mathcal{G}^M$ by considering the specification constant for $\mathcal{G}$ at a smaller scale (which depends on M). (See Corollary 5.8.)   ∎

To establish the desired growth rates on $\Lambda_R(Q, \delta, \phi)$, we need two technical counting results from [10]. These results are used implicitly in the proof of Theorem 1.1, and we do not provide a self-contained proof in the interest of concision. However, we do discuss why they hold in our setting.

As noted in Section 1.1, the conditions that we check differ slightly from those used in [10]. The only case where they are not immediately stronger conditions is the pressure estimate. In [10], the authors need to define the pressure of a discretized collection of orbit segments $P([\mathcal{P}] \cup [S], \phi) < P(\phi)$. Because we use λ-decompositions, we do not need to consider the pressure of collections of orbit segments (this is the

content of [5, Lemma 3.5, Theorem 3.6] and [6, Proposition 4.2]). Instead, it suffices to show that $P\left(\bigcap_{t\in\mathbb{R}} g_t \lambda^{-1}(0), \phi\right) < P(\phi)$, which is precisely the condition $P(\mathrm{Sing}, \phi) < P(\phi)$.

The lemmas we will use are the following.

Lemma 8.5 ([10, Lemma 4.12]).   There exist $C, \varepsilon, M > 0$ such that for all $t > 0$, there exists a (t, ε)-separated set E_t with the following properties:

- $\sum_{\gamma \in E_t} \exp\left(\int_0^t \phi(g_u \gamma)\, du\right) \geq C e^{tP(\phi)}$
- $E_t \subset \{\gamma \in GS \mid (\gamma, t) \in \mathcal{G}^M\}$.

Lemma 8.6 ([10, Lemma 4.11]).   For all $\varepsilon > 0$ sufficiently small, there exists a constant $D > 0$ such that for any (t, ε)-separated set E_t, we have

$$\sum_{\gamma \in E_t} \exp\left(\int_0^t \phi(g_u \gamma)\, \mathrm{d}u\right) \leq D e^{tP(\phi)}.$$

We are now ready to prove our growth rates.

Proposition 8.7.   For all $\delta > 0$, there exists a constant $\hat{C}$ such that

$$\Lambda_R(Q, \delta, \phi) \geq \frac{\hat{C}}{Q} e^{QP(\phi)}$$

for all sufficiently large Q.

The proof of this proposition follows almost exactly the proof of the lower bound in [4, Proposition 6.4], replacing the use of [4, Corollary 4.8] with Proposition 8.4. We include it here for completeness.

Proof.   Let C, ε, M, and E_t be as in Lemma 8.5. Now, choose $\rho < \frac{\varepsilon}{3}$ small enough that the Bowen property at scale ρ holds on $\mathcal{G}^M$ (that this is possible follows immediately from the fact that $\mathcal{G}$ has the Bowen property). Then, by Proposition 8.4, there exists $T' > 0$ so that when $t > \frac{\theta_0}{\eta} + 2M$, there is an injective mapping from E_t to a set P_t of regular closed geodesics with periods in $[t + T' - \delta, t + T']$, that is, for any $\xi \in P_t$, there exists $u \in [t + T' - \delta, t + T']$ such that $g_u \xi = \xi$. In particular, for all $\gamma \in E_t$, there exists $\xi \in P_t$ so that $d_{GS}(g_u \xi, g_u \gamma) \leq \rho$ for all $u \in [0, t]$. Because the mapping is injective and ϕ has the

Bowen property at scale ρ on $\mathcal{G}^M$, it follows from Lemma 8.5 that

$$\sum_{\xi \in P_t} \exp\left(\int_0^t \phi(g_u \xi)\, \mathrm{d}u\right) \geq Ce^{-K} e^{tP(\phi)}$$

for some constant K independent of t. Now, writing $\Phi(\xi) = \int_0^{\ell(\xi)} \phi(g_u \xi)\, \mathrm{d}u$, we can then write

$$\sum_{\xi \in P_t} \exp(\Phi(\xi)) \geq \sum_{\xi \in P_t} \exp\left(\int_0^t \phi(g_u \xi)\, \mathrm{d}u - T'\|\phi\|\right) \geq Ce^{-(K+T'\|\phi\|)} e^{tP(\phi)}.$$

At this point, we can almost relate this to $\Lambda_R(Q, \delta, \phi)$. However, there is a possibility that $\xi_1, \xi_2 \in P_t$ both represent the same closed geodesic path, that is, there exists u so that $g_u \xi_1 = \xi_2$. As P_t is (t, ρ)-separated and $d_{GS}(\eta, g_u \eta) = u$, there are at most $\frac{t+T'}{\rho}$ such repetitions. Hence, if $Q \geq T$, by setting $Q = t + T'$, we have

$$\Lambda_R(Q, \delta, \phi) \geq \left(\frac{\rho}{Q}\right) Ce^{-K} e^{-T'(\|\phi\|+P(\phi))} e^{QP(\phi)}.$$

∎

In order to see that the growth rate is not too large, we use Lemmas 8.2 and 8.6.

Proposition 8.8.   For all $\delta > 0$, there exists a constant $D > 0$ such that

$$\Lambda_R(Q, \delta, \phi) \leq De^{\delta\|\phi\|} e^{QP(\phi)}$$

for all sufficiently large Q.

Proof.   By Lemma 8.2, any set of representatives of $\mathrm{Per}_R[Q - \delta, Q]$ is (Q, ε)-separated for ε sufficiently small, and in particular, small enough to apply Lemma 8.6. Now, given $[\gamma] \in \mathrm{Per}_R[Q - \delta, Q]$, observe that $\left|\Phi(\gamma) - \int_0^Q \phi(g_u \gamma)\, \mathrm{d}u\right| \leq \delta\|\phi\|$ because we know the period of γ is at least $Q - \delta$. Consequently, it follows that for such an ε, there exists $D > 0$ such that

$$\Lambda_R(Q, \delta, \phi) \leq e^{\delta\|\phi\|} \sum_{[\gamma] \in \mathrm{Per}_R[Q-\delta, Q]} \exp\left(\int_0^Q \phi(g_u \gamma)\, \mathrm{d}u\right) \leq e^{\delta\|\phi\|} De^{QP(\phi)}.$$

∎

Proof of Theorem 8.1.   Propositions 8.7 and 8.8 imply that

$$\lim_{Q \to \infty} \frac{1}{Q} \log \Lambda_R(Q, \delta, \phi) = P(\phi).$$

By Proposition 8.3, it follows that $\lim_{Q \to \infty} \mu_{Q,\delta} = \mu$.   ∎

References

[1]   Ballmann, W. *Lectures on Spaces of Nonpositive Curvature*, vol. 25. DMV Seminar. Basel: Birkhäuser, 1999.

[2]   Bowen, R. "Some systems with unique equilibrium states." *Math. Syst. Theory* 8, no. 3 (1975): 193–202. https://doi.org/10.1007/BF01762666.

[3]   Bridson, M. R. and A. Haefliger. *Metric Spaces of Non-Positive Curvature*, vol. 319. Grundlehren der Mathematischen Wissenschaften. Berlin: Springer, 1999.

[4]   Burns, K., V. Climenhaga, T. Fisher, and D. Thompson. "Unique equilibrium states for geodesic flows in nonpositive curvatures." *Geom. Funct. Anal.* 28, no. 5 (2018): 1209–59. https://doi.org/10.1007/s00039-018-0465-8.

[5]   Call, B. and D. Thompson. "Equilibrium states for products of flows and the mixing properties of rank 1 geodesic flows." *J. Lond. Math. Soc. (2)* 105, no. 2 (2022): 794–824. https://doi.org/10.1112/jlms.12517.

[6]   Call, B. "The K-property for some unique equilibrium states in flows and homeomorphisms." *Ergodic Theory Dynam. Systems* 42 (2021): 2509–32. https://doi.org/10.1017/etds.2021.61.

[7]   Chen, D., L.-Y. Kao, and K. Park. "Properties of equilibrium states for geodesic flows over manifolds without focal points." *Adv. Math.* 380 (2021): 107564. https://doi.org/10.1016/j.aim.2021.107564.

[8]   Chen, D., L.-Y. Kao, and K. Park. "Unique equilibrium states for geodesic flows over surfaces without focal points." *Nonlinearity* 33, no. 3 (2020): 1118–55. https://doi.org/10.1088/1361-6544/ab5c06.

[9]   Climenhaga, V., G. Knieper, and K. War. "Uniqueness of the measure of maximal entropy for geodesic flows on certain manifolds without conjugate points." *Adv. Math.* 376 (2021): 107452. https://doi.org/10.1016/j.aim.2020.107452.

[10] Climenhaga, V. and D. Thompson. "Unique equilibrium states for flows and homeomorphisms with non-uniform structure." *Adv. Math.* 303 (2016): 745–99. https://doi.org/10.1016/j.aim.2016.07.029.

[11] Climenhaga, V. and D. Thompson. "Beyond Bowen's Specification Property." In *Thermodynamic Formalism*, edited by M. Pollicott and S. Vaienti, 3–82. Lecture Notes in Mathematics, vol. 2290. Cham: Springer, 2021. https://doi.org/10.1007/978-3-030-74863-0_1.

[12] Colognese, P. and M. Pollicott. "Volume Growth for Infinite Graphs and Translation Surfaces." In *Dynamics: Topology and Numbers*, edited by P. Moree, A. Pohl, L. Snoha, and T. Ward, 109–23. Contemporary Mathematics, vol. 744. Providence, Rhode Island: American Mathematical Society, 2020. ISBN 9781470451004. https://doi.org/10.1090/conm/744.

[13] Constantine, D., J.-F. Lafont, and D. Thompson. "The weak specification property for geodesic flows on CAT(-1) spaces." *Groups Geom. Dyn.* 14 (2020): 297–336. https://doi.org/10.4171/GGD/545.

[14] Constantine, D., J.-F. Lafont, and D. Thompson. "Strong symbolic dynamics for geodesic flow on CAT(−1) spaces and other metric Anosov flows." *J. Éc. polytech. Math.* 7 (2020): 201–31. https://doi.org/10.5802/jep.115.

[15] Cornfeld, I. P., S. V. Fomin, and Y. G. Sinai. *Ergodic Theory*, vol. 245. Grundlehren der Mathematischen Wissenschaften. Translated by A.B. Sossinskii. New York: Springer, 1982.

[16] Dankwart, K. "Volume entropy and the Gromov boundary of flat surfaces." (2011): preprint arXiv:1101.1795.

[17] Franco, E. "Flows with unique equilibrium states." *Amer. J. Math.* 99, no. 3 (1977): 486–514. https://doi.org/10.2307/2373927.

[18] Pesin, Y. *Dimension Theory in Dynamical Systems*. Chicago: University of Chicago Press, 1997.

[19] Ricks, R. "Flat strips, Bowen-Margulis measures, and mixing of the geodesic flow for rank one CAT(0) spaces." *Ergodic Theory Dynam. Systems* 37, no. 3 (2017): 939–70. https://doi.org/10.1017/etds.2015.78.

[20] Ricks, R. "The unique measure of maximal entropy for a compact rank one locally CAT(0) space." *Discrete Contin. Dyn. Syst.* 41, no. 2 (2021): 507–23. https://doi.org/10.3934/dcds.2020266.

[21] Walters, P. *An Introduction to Ergodic Theory*, vol. 79. New York: Springer, 1982.

[22] Wright, A. "Unique equilibrium states for flows and homeomorphisms with non-uniform structure." *EMS Surv. Math. Sci.* 2, no. 1 (2015): 63–108. https://doi.org/10.4171/EMSS/9.

[23] Zorich, A. *Flat Surfaces*. Berlin: Springer, 2006.