# THE BERNOULLI PROPERTY FOR GEODESIC FLOW ON FLAT SURFACES 

BENJAMIN CALL, DAVID CONSTANTINE, ALENA ERCHENKO, NOELLE SAWYER, AND GRACE WORK


#### Abstract

Let $S$ be a compact surface of genus $\geq 2$ equipped with a metric that is flat everywhere except at finitely many cone points with angles greater than $2 \pi$. We examine the geodesic flow on $S$ and prove local product structure for a wide class of equilibrium states. Using this, we establish the Bernoulli property for these systems.


## Contents

1. Introduction ..... 1
1.1. Previous results ..... 2
1.2. Statement of results ..... 2
1.3. An outline of the paper ..... 3
Acknowledgements ..... 3
2. Preliminaries ..... 3
2.1. Definitions ..... 3
2.2. Geometric lemmas ..... 5
2.3. Local stable and unstable sets ..... 6
2.4. Bowen brackets and their behavior ..... 9
2.5. Rectangles and flow boxes ..... 10
3. Dynamical results ..... 11
3.1. The global Bowen property ..... 11
3.2. Upper and lower Gibbs properties ..... 13
3.3. Local product structure for equilibrium states ..... 14
4. The Bernoulli property ..... 26
4.1. Definitions and lemmas ..... 26
4.2. An outline of the argument ..... 28
4.3. Steps $1 \& 2$ : Good partitions ..... 29
4.4. Step 3: Layerwise intersection ..... 32
4.5. Step 4: Local definition of $\psi$ ..... 34
4.6. Step 5: Completing the argument ..... 34
References ..... 38

## 1. Introduction

In this paper we prove that a wide class of equilibrium states for the geodesic flow on translation surfaces have local product structure. This in turn allows us to show that these equilibrium states have the Bernoulli property, an improvement on the K-property that the authors have shown previously [CCE $\left.{ }^{+} 23\right]$.

More precisely, we consider equilibrium states for potential functions which are locally constant on a neighborhood of a 'singular' set for the flow - a set of geodesics which do not experience any of the hyperbolic dynamics of the flow. Towards our main results, we prove that these potentials have the global Bowen property, an improvement from the non-uniform Bowen property previously established in $\mathrm{CCE}^{+} 23$.

In a forthcoming paper, we will show that a wide class of equilibrium states for geodesic flows on nonpositively curved manifolds have local product structure by adapting the techniques presented in the current
paper. We expect that the techniques used can be extended beyond the discussed settings to some other applications of the Climenhaga-Thompson decomposition machinery.
1.1. Previous results. The first result showing geodesic flows are Bernoulli was Ornstein and Weiss's proof that geodesic flows on compact, negatively curved surfaces are Bernoulli with respect to the Liouville measure OW73. This work builds on Ornstein's work providing the first examples of Bernoulli flows in Orn70, and provides a framework for showing a flow is Bernoulli that has frequently been followed, including in the present paper. (See section 4.2 for an outline of this argument.) Ratner noted that the Ornstein-Weiss argument could be applied to transitive, $C^{2}$ Anosov flows with Gibbs measures Rat74.

One direction in which to generalize these results is to situations with non-uniform hyperbolicity. In Pes77b, Pes77a, Pesin relaxed the negative curvature assumptions of OW73] to surfaces with no focal points. Burns and Gerber showed that even in some situations with significant positive curvature, specifically for some special metrics on 2-spheres, the geodesic flow is still Bernoulli [BG89]. As a corollary of the work of Ledrappier, Lima and Sarig LLS16, the measure of maximal entropy is Bernoulli for the geodesic flow on any closed surface whose curvature is not identically zero. Call and Thompson prove that the measure of maximal entropy is Bernoulli for closed rank-1 manifolds CT22.

Chernov and Haskell proved that non-uniformly hyperbolic flows with the $K$-property are Bernoulli CH96. Similarly, Alansari proves that the $K$-property implies Bernoulli in the setting of hyperbolic measures with local product structure Ala22.

A second direction of research relevant to our work here has been geodesic flows on non-manifold spaces. Roblin originally showed the existence of the measure of maximal entropy for geodesic flow in the CAT(-1) setting Rob03. Constantine, Lafont and Thompson proved the geodesic flow on a compact, locally CAT(-1) space is Bernoulli (or constructed from a Bernoulli shift) for equilibrium states CLT20b. In BAPP19, Broise-Alamichel, Parkonnen, and Paulin present extensive results on this setting; they significantly weaken the compactness assumption, but do need an assumption on the types of potential function allowed. Recent work in the non-compact direction but without this restriction on the potentials is due to Dilsavor and Thompson DT23.
1.2. Statement of results. Let $S$ be a compact surface of genus $g \geq 2$. Equip $S$ with a flat Riemannian metric away from a finite collection of 'cone points,' at each of which the total angle is greater than $2 \pi$. (See $\$ 2$ for precise definitions). Let $g_{t}$ be the geodesic flow on $G S$, the space of geodesics on $S$.

Given a potential function $\phi: G S \rightarrow \mathbb{R}$, one can hope to find equilibrium states for $\phi$. In our previous paper $\left[\mathrm{CCE}^{+} 23\right.$, we proved that if $\phi$ is Hölder and satisfies a pressure gap condition (see [CCE ${ }^{+} 23$, Theorem A]), then it has a unique equilibrium state $\mu$ that satisfies the $K$-property. We also show ( $\left[\mathrm{CCE}^{+} 23\right.$, Theorem B]) that potentials that are locally constant on some neighborhood of the singular set for the flow satisfy this pressure gap condition, providing a more easily checked condition on $\phi$.

In this paper we improve these results for locally constant potentials by proving more refined properties of the dynamics of $\left(G S, g_{t}, \mu\right)$.

We say a measure $\mu$ on $G S$ has local product structure if at small scales, $\mu$ is equivalent to a product measure, constructed using a (geometric) local product structure based on the unstable and weak stable foliations. (See Sections 2.3 and 3.3 for precise definitions.)

Our first main theorem is the following:
Theorem A. Let $\left(G S, \mu, g_{t}\right)$ be as above, where $\mu$ is the equilibrium state for a Hölder potential which is locally constant on some neighborhood of the singular set for $g_{t}$. For any $\varepsilon>0$ there is a subset $E$ of $G S$ with $\mu(E)>1-\varepsilon$ on which the measure $\mu$ has local product structure.

A measure-preserving flow $\left(X, \mu,\left\{\Phi_{t}\right\}\right)$ is said to be Bernoulli if for all $t \neq 0$ the discrete dynamical system $\left(X, \mu, \Phi_{t}\right)$ is isomorphic to a Bernoulli shift.

Our second main theorem is the following:
Theorem B. $\left(G S, \mu, g_{t}\right)$ as in Theorem $A$ is Bernoulli.
In the process of proving these theorems, we examine the Bowen property for the potential $\phi$ relative to our flow (see Definition 3.2). The Bowen property for some collection $\mathcal{C}$ of orbit segments says, roughly, for any two segments from $\mathcal{C}$ of equal length which stay close together, the integrals of $\phi$ along the segments are
close, uniformly over all of $\mathcal{C}$ and over all lengths of the segments. In $\mathrm{CCE}^{+} 23$ we proved this property for a large collection of 'good' segments - those that experience a definite amount of the hyperbolicity of the flow. (See Prop 3.5 for the details.)

In the present paper we improve this to a global Bowen property, where $\mathcal{C}$ can be taken to be all orbit segments. The argument uses the technology of $\lambda$-decompositions, which has been the main technical tool for applications of the Climenhaga-Thompson machinery. Hence, this argument, in particular, Proposition 3.6 may be of independent interest in other settings.

Theorem C. For $\left(G S, \mu, g_{t}\right)$ as above, and for $\phi$ which is Hölder and is locally constant on $B(\operatorname{Sing}, \delta)$ for some $\delta>0$, $\phi$ has the global Bowen property.
1.3. An outline of the paper. The outline of this paper is as follows.

In Section 2, we introduce the necessary geometric background, beginning with some basic geometric lemmas (Section 2.2) as well as developing properties of the stable and unstable sets in our setting (Section 2.3 . In Sections 2.4 and 2.5. we introduce the Bowen bracket operations and exploit the local (topological) product structure of $G S$.

In Section 3 we prove some of our main results on the dynamics of the geodesic flow. In Section 3.1, we show that the potentials considered in this paper satisfy the Bowen property globally, rather than the nonuniform version. Then, in Section 3.2 , we recall the Gibbs property developed for the Climenhaga-Thompson machinery, and show how it interacts with Bowen balls in our setting. Finally, in Section 3.3, we establish the local product structure of our unique equilibrium states.

Section 4 is devoted to the proof of the Bernoulli property. After establishing some preliminary definitions in Section 4.1, we give an outline of the steps of an argument for the Bernoulli property due to Ornstein and Weiss OW73 in Section 4.2 Section 4.3 covers the first two steps of this outline, which mainly involve recalling Lemmas from previous papers and the local product structure proved at the end of Section 3 . In Section 4.4 we prove a "layerwise" intersection property for a partition which is needed in the proof. Sections 4.5 and 4.6 conclude the proof.

## Acknowledgements

We would like to thank the American Institute of Mathematics for their hospitality and support during the SQuaRE "Thermodynamic formalism for CAT(0) spaces". We would also like to thank Vaughn Climenhaga for helpful comments regarding the presentation of the paper. AE was supported by the Simons Foundation (Grant Number 814268 via the Mathematical Sciences Research Institute, MSRI) and NSF grant DMS2247230. BC was partially supported by NSF grant DMS-2303333.

## 2. Preliminaries

In this section we collect a few preliminaries which will be used throughout the proof.
2.1. Definitions. First, we define the main objects of study.

Throughout, $S$ is a flat surface with finitely many large cone angle singularities. That is, $S$ is a compact, connected surface of genus $g \geq 2$ equipped with a metric which is flat except at finitely many points, called cone points and denoted by Con. The total angle around $p \in C$ on is denoted $\mathcal{L}(p)$ and satisfies $\mathcal{L}(p)>2 \pi$. We denote by $\theta_{0}$ the minimum of $\mathcal{L}(p)-2 \pi$ over all $p \in C$ on.
$\tilde{S}$ is the universal cover of $S$, and is a $\operatorname{CAT}(0)$ space. (See BH99, Part II] for definitions and basic results on $\operatorname{CAT}(0)$ spaces.) $d_{S}$ and $d_{\tilde{S}}$ are the metrics on $S$ and $\tilde{S}$, respectively. The unique geodesic segment joining points $\tilde{p}, \tilde{q} \in \tilde{S}$ is denoted $[\tilde{p}, \tilde{q}]$.

The geodesic flow takes place on the the space of geodesics:

$$
\begin{aligned}
& G S=\{\gamma: \mathbb{R} \rightarrow S: \gamma \text { is a local isometry }\} \\
& G \tilde{S}=\{\tilde{\gamma}: \mathbb{R} \rightarrow \tilde{S}: \tilde{\gamma} \text { is a local isometry }\}
\end{aligned}
$$

We endow these spaces with the following metrics:

$$
d_{G \tilde{S}}\left(\tilde{\gamma}_{1}, \tilde{\gamma}_{2}\right)=\int_{-\infty}^{\infty} d_{\tilde{S}}\left(\tilde{\gamma}_{1}(s), \tilde{\gamma}_{2}(s)\right) e^{-2|s|} d s
$$

$$
d_{G S}\left(\gamma_{1}, \gamma_{2}\right)=\inf _{\tilde{\gamma}_{1}, \tilde{\gamma}_{2}} d_{G \tilde{S}}\left(\tilde{\gamma}_{1}, \tilde{\gamma}_{2}\right)
$$

where the infimum is taken over all lifts of $\gamma_{1}$ and $\gamma_{2}$ to $G \tilde{S}$.
The geodesic flow is defined by $g_{t}(\gamma(s))=\gamma(s+t)$.
The geometric feature that drives the dynamical properties we study is the ability of geodesics to turn with various angles when they encounter cone points.

Definition 2.1. The turning angle of $\gamma$ at time $t$ is $\theta(\gamma, t) \in\left(-\frac{1}{2} \mathcal{L}(\gamma(t)), \frac{1}{2} \mathcal{L}(\gamma(t))\right]$ and is the signed angle between the segments $[\gamma(t-\delta), \gamma(t)]$ and $[\gamma(t), \gamma(t+\delta)]$ (for sufficiently small $\delta>0$ ). A positive (resp. negative) sign for $\theta$ corresponds to a counterclockwise (resp. clockwise) rotation with respect to the orientation of $[\gamma(t-\delta), \gamma(t)]$.

Geodesics that do not encounter cone points do not experience the hyperbolic dynamics caused by those points and hence are (from a dynamical perspective) singular:

Definition 2.2. The singular geodesics are

$$
\text { Sing }=\{\gamma \in G S:|\theta(\gamma, t)|=\pi \quad \forall t \in \mathbb{R}\}
$$

Sing is $g_{t}$-invariant, closed, and, hence, compact.
Some, but not all, singular geodesics lie in flat strips:
Definition 2.3. $A$ flat half-strip in $\tilde{S}$ is an isometric embedding of $[0, \infty) \times[a, b]$ into $\tilde{S}$ for some $[a, b]$. A flat strip is an isometric embedding of $\mathbb{R} \times[a, b]$ into $\tilde{S}$ for some $[a, b]$.

In our previous paper $\mathrm{CCE}^{+} 23$, we defined a function $\lambda: G S \rightarrow[0, \infty)$ which played an essential role in studying the dynamics of the geodesic flow. Roughly speaking, $\lambda(\gamma)$ measures the hyperbolicity of the dynamics experienced by $\gamma-$ a large value of $\lambda$ means that at some point near time zero, $\gamma$ turns with an angle significantly different from $\pi$ at a cone point. For the details of the definition of $\lambda$, see [CCE ${ }^{+} 23$, §3]. Here we note the key properties of $\lambda$ we will need:

- $\bigcap_{t \in \mathbb{R}} g_{t} \lambda^{-1}(0)=\operatorname{Sing}\left(\left[\mathrm{CCE}^{+} 23\right.\right.$, Cor 3.5$\left.]\right)$.
- $\lambda$ is lower semicontinuous $\left(\left[\mathrm{CCE}^{+} 23\right.\right.$, Lemma 3.8]).
- Let $s_{0}$ be a positive parameter in the definition of $\lambda$. We recall that it depends only on the geometry of $S$, specifically the minimum distance between cone points in $S$. Then, the following holds.

Lemma 2.4. ( $\left.\mathrm{CCE}^{+} 23\right]$, Proposition 3.9) If $\lambda(x)>c>0$, then either:
(1) There exists $t \in\left(-s_{0}, s_{0}\right)$ such that $x$ turns at $t$ with angle at least $c_{0}$, or
(2) There exist $t_{1} \in\left(-\frac{\theta_{0}}{2 c},-s_{0}\right]$ and $t_{2} \in\left[s_{0}, \frac{\theta_{0}}{2 c}\right)$ such that $x$ first turns at $t_{1}$ and $t_{2}$ with angle at least $c s_{0}$, where $\theta_{0}$ depends only on the geometry of $S$.

Geodesics not in Sing are regular, and the value of $\lambda$ measures their regularity:
Definition 2.5. For all $c>0$, let

$$
\operatorname{Reg}(c):=\{\gamma \in G S \mid \lambda(\gamma) \geq c\}
$$

As we will see below, for many of our arguments about geodesic segments, only their level of regularity at the beginning and end are important.

Definition 2.6. Let $\mathcal{C}(c)=\left\{(x, t) \in G S \times \mathbb{R} \mid \lambda(x) \geq c\right.$ and $\left.\lambda\left(g_{t} x\right) \geq c\right\}$ be the set of orbit segments with large values of $\lambda$ at their beginning and end.
$\partial_{\infty} \tilde{S}$ is the boundary at infinity of $\tilde{S}$ (see BH99, §II.8]). If $\tilde{\gamma} \in G \tilde{S}$, then $\tilde{\gamma}( \pm \infty)$ are its forward and backwards endpoints at infinity. Geodesics that share an endpoint at infinity either bound a flat half-strip or are asymptotic in the following sense:

Definition 2.7. We say that geodesics $\tilde{\gamma}_{1}, \tilde{\gamma}_{2} \in \tilde{S}$ are $*$-cone point asymptotic, where $* \in\{+,-\}$, if $\tilde{\gamma}_{1}(* \infty)=\tilde{\gamma}_{2}(* \infty)=\xi$ and there is a cone point $p$ such that the ray defined by $p$ and $\xi$ is contained in both $\tilde{\gamma}_{1}$ and $\tilde{\gamma}_{2}$.
2.2. Geometric lemmas. It is useful to relate $d_{G S}$ to $d_{S}$, the metric on the surface itself. First, if two geodesics are close in $G S$, then they are close in $S$ at time zero.

Lemma 2.8 (CLT20a, Lemma 2.8). For all $\gamma_{1}, \gamma_{2} \in G S$,

$$
d_{S}\left(\gamma_{1}(0), \gamma_{2}(0)\right) \leq 2 d_{G S}\left(\gamma_{1}, \gamma_{2}\right)
$$

Furthermore, for $s, t \in \mathbb{R}, d_{S}\left(\gamma_{1}(s), \gamma_{2}(t)\right) \leq 2 d_{G S}\left(g_{s} \gamma_{1}, g_{t} \gamma_{2}\right)$.

Conversely, if two geodesics are close in $S$ for a significant interval of time surrounding zero, then they are close in $G S$ :

Lemma 2.9 (CLT20a, Lemma 2.11). Let $\varepsilon$ be given and $a<b$ arbitrary. There exists $T=T(\varepsilon)>0$ such that if $d_{S}\left(\gamma_{1}(t), \gamma_{2}(t)\right)<\varepsilon / 2$ for all $t \in[a-T, b+T]$, then $d_{G S}\left(g_{t} \gamma_{1}, g_{t} \gamma_{2}\right)<\varepsilon$ for all $t \in[a, b]$. For small $\varepsilon$, we can take $T(\varepsilon)=-\log (\varepsilon)$.

We will need a few refinements of Lemma 2.9 throughout the paper, which we record here.

Lemma $2.10\left(\mathrm{CCE}^{+} 23\right]$, Lemma 2.13). Suppose that $d_{S}\left(\gamma_{1}(t), \gamma_{2}(t)\right)=0$ for all $t \in[a, b]$. Then, for all $t \in[a, b], d_{G S}\left(g_{t} \gamma_{1}, g_{t} \gamma_{2}\right) \leq e^{-2 \min \{|t-a|,|t-b|\}}$.

For $t>0$, the time- $t$ geodesic flow is $e^{2 t}$-Lipschitz (see, e.g., [CEE ${ }^{+} 23$, Lemma 2.14]). The next two lemmas give similar results in specific geometric situations.

Lemma 2.11. Suppose that $\gamma(t)=\eta(t)$ for all $t \leq r$ for some $r \in \mathbb{R}$. Then there exists $C:=C(r)>0$ such that $d_{G S}\left(g_{t} \gamma, g_{t} \eta\right) \leq C e^{2 t} d_{G S}(\gamma, \eta)$ for $t \leq 0$. If $r \geq 0$, then $C=1$.

Proof. By definition, for appropriately chosen lifts $\tilde{\gamma}$ and $\tilde{\eta}$,

$$
\begin{aligned}
d_{G S}\left(g_{t} \gamma, g_{t} \eta\right) & =\int_{-\infty}^{\infty} e^{-2|s|} d_{\tilde{S}}(\tilde{\gamma}(s+t), \tilde{\eta}(s+t)) d s \\
& =\int_{-\infty}^{\infty} e^{-2|s-t|} d_{\tilde{S}}(\tilde{\gamma}(s), \tilde{\eta}(s)) d s
\end{aligned}
$$

By our assumption on $\gamma$ and $\eta$,

$$
\begin{aligned}
& =\int_{r}^{\infty} e^{-2|s-t|} d_{\tilde{S}}(\tilde{\gamma}(s), \tilde{\eta}(s)) d s \\
& \leq e^{\max \{-4 r, 0\}} e^{2 t} \int_{r}^{\infty} e^{-2|s|} d_{\tilde{S}}(\tilde{\gamma}(s), \tilde{\eta}(s)) d s \\
& =e^{\max \{-4 r, 0\}} e^{2 t} d_{G S}(\gamma, \eta)
\end{aligned}
$$

Lemma 2.12. Suppose that $\gamma(t)=\eta(t+\delta)$ for all $t \geq r$ for some $r \in \mathbb{R}$. Then there exists $D:=D(r, \delta)>0$ such that $d_{G S}\left(g_{t} \gamma, g_{t} \eta\right) \leq D e^{-2 t} d_{G S}(\gamma, \eta)+2 \delta$ for $t \geq 0$.

Proof. Observe that there exist lifts $\tilde{\gamma}, \tilde{\eta}$ of $\gamma$ and $\eta$ such that

$$
\begin{aligned}
d_{G S}\left(g_{t} \tilde{\gamma}, g_{t} \tilde{\eta}\right) & =\int_{-\infty}^{\infty} e^{-2|s|} d_{\tilde{S}}(\tilde{\gamma}(s+t), \tilde{\eta}(s+t)) d s \\
& =\int_{-\infty}^{\infty} e^{-2|s-t|} d_{\tilde{S}}(\tilde{\gamma}(s), \tilde{\eta}(s)) d s \\
& \leq \int_{-\infty}^{\infty} e^{-2|s-t|} d_{\tilde{S}}(\tilde{\gamma}(s), \tilde{\eta}(s+\delta)) d s+\int_{-\infty}^{\infty} e^{-2|s-t|} d_{\tilde{S}}(\tilde{\eta}(s+\delta), \tilde{\eta}(s)) d s \\
& =\int_{-\infty}^{\infty} e^{-2|s-t|} d_{\tilde{S}}(\tilde{\gamma}(s), \tilde{\eta}(s+\delta)) d s+\int_{-\infty}^{\infty} e^{-2|s-t|} \delta d s \\
& =\int_{-\infty}^{r} e^{-2|s-t|} d_{\tilde{S}}(\tilde{\gamma}(s), \tilde{\eta}(s+\delta)) d s+\delta \\
& \leq \int_{-\infty}^{r} e^{-2|s-t|} d_{\tilde{S}}(\tilde{\gamma}(s+\delta), \tilde{\eta}(s+\delta)) d s+2 \delta \\
& =\int_{-\infty}^{r+\delta} e^{-2|s-t-\delta|} d_{\tilde{S}}(\tilde{\gamma}(s), \tilde{\eta}(s)) d s+2 \delta \\
& \leq e^{\max \{4 r, 0\}} e^{2|\delta|} e^{-2 t} \int_{-\infty}^{r+\delta} e^{-2|s|} d_{\tilde{S}}(\tilde{\gamma}(s), \tilde{\eta}(s)) d s+2 \delta \\
& =e^{\max \{4 r, 0\}} e^{2|\delta|} e^{-2 t} d_{G \tilde{S}}(\tilde{\gamma}, \tilde{\eta})+2 \delta .
\end{aligned}
$$

Note that in the sense of Definition 2.7, the geodesics in Lemma 2.11 are (a special case of) --cone point asymptotic, and those in Lemma 2.12 are +-cone point asymptotic.
2.3. Local stable and unstable sets. The local stable and unstable sets, analogues of the local stable and unstable manifolds from the smooth setting, play an essential role in our arguments. We define them here using Busemann functions; the definition given agrees with the usual one in the smooth setting.
Definition 2.13. For a geodesic $\tilde{\gamma} \in G \tilde{S}$, the Busemann function determined by $\tilde{\gamma}$ is

$$
B_{\tilde{\gamma}}(\cdot):=B_{\tilde{\gamma}(0)}(\cdot, \tilde{\gamma}(+\infty)),
$$

where $B_{\tilde{\gamma}(0)}(\cdot, \tilde{\gamma}(+\infty)): \tilde{S} \rightarrow \mathbb{R}$ is the function

$$
x \mapsto \lim _{t \rightarrow \infty} d_{\tilde{S}}(x, \tilde{\gamma}(t))-t .
$$

Using this, we can define natural analogues of the local stable and unstable manifolds.
Definition 2.14. We define the local strong unstable set at $\gamma \in G S$ by

$$
W^{u}(\gamma, \delta)=\pi\left\{\tilde{\eta} \in G \tilde{S} \mid \tilde{\eta}(-\infty)=\tilde{\gamma}(-\infty), B_{-\tilde{\gamma}}(\tilde{\eta}(0))=0, \text { and } d_{G \tilde{S}}(\tilde{\eta}, \tilde{\gamma})<\delta\right\},
$$

where $\pi: G \tilde{S} \rightarrow G S$ is projection.
The strong unstable set at $\gamma \in G S$ is

$$
W^{u}(\gamma)=\pi\left\{\tilde{\eta} \in G \tilde{S} \mid \tilde{\eta}(-\infty)=\tilde{\gamma}(-\infty), B_{-\tilde{\gamma}}(\tilde{\eta}(0))=0\right\} .
$$

The (local) strong stable set is defined similarly, replacing $-\infty$ with $+\infty$ and $-\tilde{\gamma}$ with $\tilde{\gamma}$. We will also need the local center stable set, which is defined to allow some "mismatch of times".
Definition 2.15. The local center stable set at $\gamma$ is defined by

$$
W^{c s}(\gamma, \delta)=\pi\left\{\tilde{\eta} \in G \tilde{S} \mid \tilde{\eta}(+\infty)=\tilde{\gamma}(+\infty) \text { and } d_{G \tilde{S}}(\tilde{\eta}, \tilde{\gamma})<\delta\right\} .
$$

The stable and unstable sets are invariant under the geodesic flow. We prove this here for the local strong unstable, as we will need it later. Similar proofs for the stable and center stable can easily be obtained in the same way. We begin with some comparison geometry lemmas.
Lemma 2.16. Let $\tilde{\gamma}, \tilde{\eta} \in G \tilde{S}$ with $\tilde{\eta}(+\infty)=\tilde{\gamma}(+\infty)=\xi$. Fix $\tau>0$. Then,

$$
\left|d_{\tilde{S}}(\tilde{\eta}(0), \tilde{\gamma}(t))-\left(d_{\tilde{S}}(\tilde{\eta}(0), \tilde{\eta}(\tau))+d_{\tilde{S}}(\tilde{\eta}(\tau), \tilde{\gamma}(t))\right)\right| \rightarrow 0 \quad \text { as } \quad t \rightarrow \infty .
$$

Proof. Since $\tilde{\eta}(+\infty)=\tilde{\gamma}(+\infty)$, there exists $C>0$ such that $d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\eta}(t)) \leq C$ for all $t \geq 0$. Consider a geodesic triangle $\Delta$ with vertices $\tilde{\eta}(0), \tilde{\eta}(t), \tilde{\gamma}(t)$ for $t \gg \tau$. Pick a point $p$ on the side $[\tilde{\eta}(0), \tilde{\gamma}(t)]$ such that

$$
\frac{d_{\tilde{S}}(\tilde{\eta}(0), p)}{d_{\tilde{S}}(\tilde{\eta}(0), \tilde{\gamma}(t))}=\frac{\tau}{t}
$$

By the properties of similar triangles for the comparison Euclidean triangle for $\Delta$ and the fact that $\tilde{S}$ is CAT(0),

$$
\begin{equation*}
d_{\tilde{S}}(\tilde{\eta}(\tau), p) \leq C \frac{\tau}{t} \tag{1}
\end{equation*}
$$

Then, using the triangle inequality for various triangles,
$\left[d_{\tilde{S}}(\tilde{\eta}(0), \tilde{\eta}(\tau))-d_{\tilde{S}}(\tilde{\eta}(\tau), p)\right]+\left[d_{\tilde{S}}(\tilde{\eta}(\tau), \tilde{\gamma}(t))-d_{\tilde{S}}(\tilde{\eta}(\tau), p)\right] \leq d_{\tilde{S}}(\tilde{\eta}(0), \tilde{\gamma}(t)) \leq d_{\tilde{S}}(\tilde{\eta}(0), \tilde{\eta}(\tau))+d_{\tilde{S}}(\tilde{\eta}(\tau), \tilde{\gamma}(t))$
Combining the above inequality with (11) gives the statement.
Lemma 2.17. Let $\tilde{\gamma}, \tilde{\eta} \in G \tilde{S}$ with $\tilde{\eta}(+\infty)=\tilde{\gamma}(+\infty)$. If $B_{\tilde{\gamma}}(\tilde{\eta}(0))=0$ then $B_{g_{\tau} \tilde{\gamma}}\left(g_{\tau} \tilde{\eta}(0)\right)=0$ for all $\tau$.
Proof. Using Definition 2.13 ,

$$
\begin{aligned}
0 & =B_{\tilde{\gamma}}(\tilde{\eta}(0))=\lim _{t \rightarrow \infty}\left(d_{\tilde{S}}(\tilde{\eta}(0), \tilde{\gamma}(t))-t\right) \\
& =\lim _{t \rightarrow \infty}\left(d_{\tilde{S}}(\tilde{\eta}(0), \tilde{\eta}(\tau))+d_{\tilde{S}}(\tilde{\eta}(\tau), \tilde{\gamma}(t))-t\right) \quad \text { by Lemma } 2.16 \\
& =\lim _{t \rightarrow \infty}\left(\tau+d_{\tilde{S}}\left(g_{\tau} \tilde{\eta}(0), g_{\tau} \tilde{\gamma}(t-\tau)\right)-t\right) \\
& =\lim _{u \rightarrow \infty}\left(d_{\tilde{S}}\left(g_{\tau} \tilde{\eta}(0), g_{\tau} \tilde{\gamma}(u)\right)-u\right)=B_{g_{\tau} \tilde{\gamma}}\left(g_{\tau} \tilde{\eta}(0)\right) .
\end{aligned}
$$

Corollary 2.18. Let $\tilde{\gamma}, \tilde{\eta} \in G \tilde{S}$ with $\tilde{\eta}(-\infty)=\tilde{\gamma}(-\infty)$. If $B_{-\tilde{\gamma}}(\tilde{\eta}(0))=0$ then $B_{g_{\tau}(-\tilde{\gamma})}\left(g_{\tau} \tilde{\eta}(0)\right)=0$ for all $\tau$.

Corollary 2.19. Let $\gamma \in G S$. Then

$$
g_{\tau}\left(W^{u}(\gamma, \delta)\right) \subset W^{u}\left(g_{\tau} \gamma, e^{2 \tau} \delta\right) \quad \text { for } \quad \tau>0
$$

and

$$
g_{\tau}\left(W^{u}(\gamma, \delta)\right) \subset W^{u}\left(g_{\tau} \gamma, \delta\right) \quad \text { for } \quad \tau \leq 0
$$

Proof. If $\tilde{\eta}(-\infty)=\tilde{\gamma}(-\infty)$ then $g_{\tau} \tilde{\eta}(-\infty)=g_{\tau} \tilde{\gamma}(-\infty)$. Moreover, by Corollary 2.18, if $B_{-\tilde{\gamma}}(\tilde{\eta}(0))=0$ then $B_{g_{\tau}(-\tilde{\gamma})}\left(g_{\tau} \tilde{\eta}(0)\right)=0$. To complete the proof, we just need to bound the distance from $g_{\tau} \tilde{\gamma}$.

Let $\gamma \in G S$, and let $\eta \in W^{u}(\gamma, \delta)$. Since $\tilde{\gamma}(-\infty)=\tilde{\eta}(-\infty)$, we know that $d_{G \tilde{S}}(\tilde{\gamma}(t), \tilde{\eta}(t))$ remains bounded as $t \rightarrow-\infty$. By convexity of the distance function, it is decreasing as $t \rightarrow-\infty$ and bounded above by $\delta$ for $t \leq 0$. Therefore, for $\tau \leq 0$, we have

$$
\begin{aligned}
d_{G \tilde{S}}\left(g_{\tau} \tilde{\eta}, g_{\tau} \tilde{\gamma}\right) & =\int_{-\infty}^{\infty} e^{-2|t|} d_{\tilde{S}}(\tilde{\eta}(t+\tau), \tilde{\gamma}(t+\tau)) d t \\
& \leq \int_{-\infty}^{\infty} e^{-2|t|} d_{\tilde{S}}(\tilde{\eta}(t), \tilde{\gamma}(t)) d t \\
& =d_{G \tilde{S}}(\tilde{\eta}, \tilde{\gamma})
\end{aligned}
$$

Hence,

$$
g_{\tau}\left(W^{u}(\gamma, \delta)\right) \subset W^{u}\left(g_{\tau} \gamma, \delta\right) \quad \text { for } \quad \tau \leq 0
$$

Finally, if $\tau \geq 0$, using that $g_{\tau}$ is $e^{2 \tau}$-Lipschitz,

$$
d_{G S}\left(g_{\tau} \eta, g_{\tau} \gamma\right) \leq e^{2 \tau} d_{G S}(\eta, \gamma)<e^{2 \tau} \delta
$$

Thus, we obtain the desired statement.
The following lemmas and corollaries make the relationship between our systems and nonuniformly hyperbolic systems evident.

Lemma 2.20. Assume $\gamma \in \operatorname{Reg}(c)$. Then there exists $\delta$ such that if $\tilde{\eta} \in W^{u}(\tilde{\gamma}, \delta)$, then $\tilde{\gamma}$ and $\tilde{\eta}$ are --cone point asymptotic, and if $\tilde{\eta} \in W^{c s}(\tilde{\gamma}, \delta)$, then $\tilde{\gamma}$ and $\tilde{\eta}$ are + -cone point asymptotic.

Proof. Since $\gamma \in \operatorname{Reg}(c)$, by $\left[\mathrm{CEE}^{+} 23\right.$, Proposition 3.9], $\tilde{\gamma}$ turns with angle greater than $\pi$ at some $\tilde{p} \in \tilde{S}$. There exists $\delta>0$ small enough so that $\tilde{\eta}$ passes through the cone point $\tilde{p}\left[\mathrm{CCE}^{+} 23\right.$, Figure 2]. Now, BH99, Proposition 8.2] tells us that there are unique geodesic rays from $\tilde{p}$ that are asymptotic to $\tilde{\gamma}(+\infty)$ and $\tilde{\gamma}(-\infty)$. This completes our proof.

Lemmas 2.20 and 2.11 combine to give us the following corollary.
Corollary 2.21. For all $c>0$, there exist $\delta, C>0$ such that for all $\gamma \in \operatorname{Reg}(c)$, we have the following property. For all $\eta \in W^{u}(\gamma, \delta)$, for all $t \leq 0$,

$$
d_{G S}\left(g_{t} \gamma, g_{t} \eta\right) \leq C e^{2 t} d_{G S}(\gamma, \eta)
$$

Similarly, Lemmas 2.20 and 2.12 combine to give us an analogous corollary on center stable sets.
Corollary 2.22. For all $c>0$, there exists $\delta_{0}, D>0$ such that for all $\gamma \in \operatorname{Reg}(c)$, we have the following property. For all $\eta \in W^{c s}(\gamma, \delta)$ with $0<\delta<\delta_{0}$, for all $t \geq 0$,

$$
d_{G S}\left(g_{t} \gamma, g_{t} \eta\right) \leq D \delta
$$

We noted above that the function $\lambda: G S \rightarrow[0, \infty)$ is lower semi-continuous. On local strong unstable sets we in fact have full continuity.
Proposition 2.23. Let $\lambda: G S \rightarrow[0,+\infty)$ be as in $\mathrm{CCE}^{+} 23$, Definition 3.3] with parameter $s_{0}$ and $\gamma \in G S$. Then for all $\varepsilon>0$ there exists $\delta>0$ such that for all $\xi \in W^{u}(\gamma, \delta)$, we have $|\lambda(\gamma)-\lambda(\xi)|<\varepsilon$.

Proof. Assume that $\gamma \in \operatorname{Reg}(c)$. Then, we have two possibilities:
(1) There exists $u \in\left(-s_{0}, s_{0}\right)$ so that $\gamma$ turns with angle larger than $\pi$ at the cone point $\gamma(u)$.

In this case, $\lambda$ is continuous at $\gamma$ by Case 1 in $\mathrm{CCE}^{+} 23$, Proof of Lemma 3.8] so we have the statement if we only consider $\xi \in W^{u}(\gamma, \delta)$.
(2) There exist $u_{1} \in\left[-\frac{\theta_{0}}{2 c},-s_{0}\right]$ and $u_{2} \in\left[s_{0}, \frac{\theta_{0}}{2 c}\right]$ such that $\gamma$ has turning angles at least $c s_{0}$ away from $\pm \pi$ at $\gamma\left(u_{1}\right)$ and $\gamma\left(u_{2}\right)$.

Following Case 2 in $\mathrm{CCE}^{+} 23$, Proof of Lemma 3.8], for sufficiently small $\delta$ (depending on $\varepsilon$ and $S), \gamma$ and $\xi$ share the segment $\gamma\left[u_{1}, u_{2}\right]$. Moreover, since $\xi \in W^{u}(\gamma, \delta)$, they actually share $\gamma\left[-\infty, u_{2}\right]$ and $\gamma(0)=\xi(0)$. Thus, for such a choice of $\delta, \gamma$ and $\xi$ meet cone points with turning angle larger than $\pi$ at the same time and turn there with close angles as can be seen in Case 2 of $\mathrm{CCE}^{+} 23$, Proof of Lemma 3.8]. Thus, $|\lambda(\gamma)-\lambda(\xi)|<\varepsilon$.
Assume that $\lambda(\gamma)=0$. Then, by $\mathrm{CCE}^{+} 23$, Proposition 3.4], either $\lambda\left(g_{t} \gamma\right)=0$ for all $t \leq 0$ or for all $t \geq 0$. Therefore, we have the following possibilities:
(1) $\lambda\left(g_{t} \gamma\right)=0$ for all $t \leq 0$ (so $\gamma$ does not turn with angle larger than $\pi$ on $\gamma[-\infty, 0]$ ), and there exists $u>0$ such that $\gamma$ turns with angle larger than $\pi$ at $\gamma(u)$ and has turning angles $\pm \pi$ on $\gamma[0, u)$.

In this case, for sufficiently small $\delta$ (depending on $\gamma$ and $\varepsilon$ ), $\xi \in W^{u}(\gamma, \delta)$ has to pass through $\gamma(u)$ and turn with angle close to the turning angle of $\gamma$ at $\gamma(u)$ by $\mathrm{CCE}^{+} 23$, Lemmas 2.11 and 2.14]. Moreover, since $\xi(-\infty)=\gamma(-\infty)$ and $B_{-\tilde{\gamma}}(\tilde{\xi}(0))=0$, we have $\xi(t)=\gamma(t)$ for $t \leq u$. Thus, $\gamma$ and $\xi$ see the same first cone point at the same time and turn with close angles so $\lambda(\xi)$ and $\lambda(\gamma)$ are close.
(2) $\lambda\left(g_{t} \gamma\right)=0$ for all $t \geq 0$ (so $\gamma$ does not turn with angle larger than $\pi$ on $\gamma[0, \infty]$ ), and there exists $u<0$ such that $\gamma$ turns with angle larger than $\pi$ at $\gamma(u)$ and has turning angles $\pm \pi$ on $\gamma(u, 0]$.

By choosing $\delta$ small enough (depending on $\gamma$ and $\varepsilon$ ), we can guarantee that for $\xi \in W^{u}(\gamma, \delta)$ we have that $\xi(t)=\gamma(t)$ for $t \leq u$ and $\xi$ does not turn with angle larger than $\pi$ on $\xi\left(u, 2 \max \left\{\frac{\theta_{0}}{\lambda^{s s}(\gamma)}, \frac{\theta_{0}}{\varepsilon}\right\}\right]$. Then, $|\lambda(\xi)-\lambda(\gamma)|<\varepsilon$.
(3) $\lambda\left(g_{t} \gamma\right)=0$ for all $t$ (so $\gamma$ does not turn with angle larger than $\pi$ at any point).

If $\tilde{\gamma}$ is in the interior of a flat strip of $\tilde{S}$, then for sufficiently small $\delta, \tilde{\xi}$ is also in the interior of a flat strip of $\tilde{S}$ for $\xi \in W^{u}(\gamma, \delta)$. Thus, $\lambda(\xi)=0$.

If $\tilde{\gamma}$ is not in the interior of a flat strip, then $\tilde{\gamma}$ passes through cone points but turns with angle $\pm \pi$ at those points. If $\lambda(\xi)=0$ there is nothing to show. If there exists $u>0$ such that $\gamma(u)$ is a cone point, then for sufficiently small $\delta, \xi(t)=\gamma(t)$ for $t \leq u$ so $\lambda(\xi)=0$. Assume that $\lambda(\xi)>0$ and $\gamma(0,+\infty)$ does not contain cone points. Let $u \leq 0$ be the closest number to 0 such that $\gamma(u)$
is a cone point. Then, by taking sufficiently small $\delta$, we can ensure that $\xi(t)=\gamma(t)$ for $t \leq u$ and $\xi$ turns with a small angle at $\xi(u)$. Specifically, we can ensure that $\frac{\theta(\xi, u)}{\max \left\{s_{0},|u|\right\}}$ is less than $\varepsilon$. Then, $|\lambda(\xi)-\lambda(\gamma)|<\varepsilon$.
Corollary 2.24. Let $\lambda: G S \rightarrow[0,+\infty)$ be as in $\left[\mathrm{CCE}^{+} 23\right.$, Definition 3.3] and $\gamma \in G S$. Then, for any $a>0$ and $t \in \mathbb{R},\left.\lambda \circ g_{t}\right|_{W^{u}(\gamma, a)}$ is continuous.

Proof. Let $a>0$ and $t \in \mathbb{R}$. Then $g_{t} W^{u}(\gamma, a) \subset W^{u}\left(g_{t} \gamma, e^{2 t} a\right)$ by Corollary 2.19. Consequently, $\left.\lambda\right|_{g_{t} W^{u}(\gamma, a)}$ is continuous by Proposition 2.23 .

### 2.4. Bowen brackets and their behavior.

Definition 2.25. Let $\delta>0$. Consider $\gamma, \eta \in \operatorname{Reg}$ such that $d_{G S}(\gamma, \eta) \leq \delta$. We define the Bowen bracket as

$$
[\gamma, \eta]:=W^{c s}(\gamma, \delta) \cap W^{u}(\eta, \delta)
$$

Definition 2.26. Let $\delta>0$. Consider $\gamma, \eta \in \operatorname{Reg}$ such that $d_{G S}(\gamma, \eta) \leq \delta$. We define the su-Bowen bracket as

$$
\langle\gamma, \eta\rangle:=W^{s}(\gamma, \delta) \cap W^{u}(\eta, \delta)
$$

Remark. Note that if either of these brackets contains a regular geodesic, then the bracket consists of at most one point, as otherwise, we would have two geodesics $\xi_{1}, \xi_{2}$ with $\xi_{1}( \pm \infty)=\xi_{2}( \pm \infty)$. This is only possible if $\xi_{1}=g_{t} \xi_{2}$ for some $t$; however, this would contradict both $\xi_{1}$ and $\xi_{2}$ belonging to $W^{u}(\eta, \delta)$.

A priori, $[\gamma, \eta]$ and $\langle\gamma, \eta\rangle$ might be empty. But using our $\lambda$, we can precisely describe situations where these brackets exist and consist of a single, easily described geodesic.

Lemma 2.27. Let $c>0$. Suppose $x$ is chosen with $\lambda\left(g_{-s_{0}} x\right), \lambda\left(g_{s_{0}} x\right)>c$. Then there exists $\delta:=\delta(c)>0$ such that if $\gamma \in W^{u}(x, \delta), \eta \in W^{s}(x, \delta)$, then $\langle\gamma, \eta\rangle=\xi$, where $\xi(r)=\gamma(r)$ for $r \geq 0$ and $\xi(r)=\eta(r)$ for $r \leq 0$.
Proof. First, if $\lambda\left(g_{s_{0}} x\right), \lambda\left(g_{-s_{0}} x\right)>c$, then the first cone points $x$ turns at are at times $-2 s_{0}-\frac{\theta_{0}}{2 c}<t_{1}<0$ and $0<t_{2}<2 s_{0}+\frac{\theta_{0}}{2 c}$ and with angle at least $s_{0} c$ by Lemma 2.4 Then choose $\delta$ small enough so that given $y$ with $d_{G S}(y, x) \leq \delta$, then $y$ turns at $x\left(t_{1}\right)$ and $x\left(t_{2}\right)$ with angle at least $\frac{s_{0} c}{2}$. In particular, if $\gamma \in W^{u}(x, \delta)$ and $\eta \in W^{s}(x, \delta)$, this implies that $\gamma(r)=\eta(r)=x(r)$ for $r \in\left[t_{1}, t_{2}\right]$, which in turn guarantees that $\xi$ as constructed is a geodesic. Furthermore, we have $d_{G S}(\xi, \gamma)=d_{G S}(\eta, x)$, and similarly $d_{G S}(\xi, \eta)=d_{G S}(\gamma, x)$. Finally, by our choice of $\delta,\langle\gamma, \eta\rangle$ consists of a unique geodesic, because $\xi$ is bounded away from a flat strip.

Corollary 2.28. Assume we are in the setting of Lemma 2.27. Then, $d_{G S}(\xi, x)=d_{G S}(x, \gamma)+d_{G S}(x, \eta)$.
Proof. Straightforward from the definition of $d_{G S}$ and $\xi$.
Corollary 2.29. Assume we are in the setting of Lemma 2.27. Given $\xi=\langle\gamma, \eta\rangle$, then $\lambda\left(g_{r} \xi\right)>\frac{c s_{0}}{2\left(3 s_{0}+\frac{\theta_{0}}{2 c}\right)}$ for $-s_{0} \leq r \leq s_{0}$
Proof. Recall that by our choice of $\delta, \xi$ first turns at a cone point at times $-\left(2 s_{0}+\frac{\theta_{0}}{2 c}\right)<t_{1}<0<t_{2}<2 s_{0}+\frac{\theta_{0}}{2 c}$, with turning angle at least $\frac{c s_{0}}{2}$. Now, observe that since $\xi$ turns with angle at least $\frac{c s_{0}}{2}$ at both $t_{1}$ and $t_{2}$, we have

$$
\lambda(\xi) \geq \frac{\frac{c s_{0}}{2}}{\max \left\{\left|t_{1}\right|,\left|t_{2}\right|, s_{0}\right\}} \geq \frac{c s_{0}}{2\left(2 s_{0}+\frac{\theta_{0}}{2 c}\right)}
$$

If we let $r \in\left[-s_{0}, s_{0}\right]$ and consider $g_{r} \xi$, there are three possibilities. We could have $t_{1}-r<0<t_{2}-r$, in which case $g_{r} \xi$ first turns at a cone point at times $t_{1}-r<0<t_{2}-r$, and it turns with angle at least $\frac{c s_{0}}{2}$, or $t_{1}-r \geq 0$ or $t_{2}-r \leq 0$. In the first case, then

$$
\lambda\left(g_{r} \xi\right) \geq \frac{c s_{0}}{2 \max \left\{\left|t_{1}-r\right|,\left|t_{2}-r\right|, s_{0}\right\}} \geq \frac{c s_{0}}{2\left(2 s_{0}+\frac{\theta_{0}}{2 c}+r\right)}
$$

In the second case, $t_{1}-r \in\left[0, s_{0}\right)$, and so $g_{r} \xi$ turns at a cone point with angle at least $\frac{c s_{0}}{2}$ at some time less than $s_{0}$ away from 0 . Thus, $\lambda\left(g_{r} \xi\right)>\frac{c}{2}$. The analysis for the third case is similar.
2.5. Rectangles and flow boxes. We now exploit the local topological product structure of $G S$ to build rectangles and flow boxes. A rectangle is a transversal to the flow direction with 'rectangular' structure given by stable and unstable directions, and a flow box is a rectangle thickened in the flow direction by a small amount. We now construct these sets at small scale around regular geodesics and prove a few properties of their geometry.

Definition 2.30. Let $x$ and $\delta$ be as in Lemma 2.27. We then define an su-rectangle centered at $x$ with $\operatorname{diam}\left(R_{x}(\varepsilon)\right)<4 \varepsilon<\delta$ in the following way:

$$
\begin{equation*}
R_{x}(\varepsilon)=\left\{\langle\gamma, \eta\rangle \mid \gamma \in W^{u}(x, \varepsilon), \eta \in W^{s}(x, \varepsilon)\right\} \tag{2}
\end{equation*}
$$

Notice that for $\varepsilon<s_{0}$ (in particular, smaller than the injectivity radius of $S$ ) we have $R_{x}(\varepsilon)$ is a transversal to the flow. That is, for any $\gamma \in G S$ within $\epsilon$ of $R_{x}(\varepsilon), g_{[-5 \epsilon, 5 \epsilon]} \gamma \cap R_{x}(\varepsilon)$ is either empty or a single point.
Lemma 2.31. For $\varepsilon<\delta / 4$, the su-Bowen bracket is defined and equal to a single geodesic on any pair of geodesics in $R_{x}(\varepsilon)$.
Proof. Let $y, z \in R_{x}(\varepsilon)$. Then, $y=\left\langle\gamma_{y}, \eta_{y}\right\rangle$ and $z=\left\langle\gamma_{z}, \eta_{z}\right\rangle$ where $\gamma_{y}, \gamma_{z} \in W^{u}(x, \varepsilon)$ and $\eta_{y}, \eta_{z} \in W^{s}(x, \varepsilon)$. By Corollary 2.28 $d_{G S}(x, y), d_{G S}(x, z)<2 \varepsilon<\delta$ and so $y(r)=z(r)=x(r)$ for $r \in\left[t_{1}, t_{2}\right]$ for $t_{1}$, $t_{2}$ as in Lemma 2.27. Thus, $\langle y, z\rangle=\left\langle\gamma_{y}, \eta_{z}\right\rangle \in R_{x}(\varepsilon)$.
Lemma 2.32. Given a rectangle $R_{x}(\varepsilon)$, there exists $t_{R}:=t_{R}(c)$ such that for all $\langle\gamma, \eta\rangle \in R_{x}(\varepsilon)$ and $t \geq t_{R}$,

$$
\lambda\left(g_{t}\langle\gamma, \eta\rangle\right)=\lambda\left(g_{t} \gamma\right) \text { and } \lambda\left(g_{-t}\langle\gamma, \eta\rangle\right)=\lambda\left(g_{-t} \eta\right)
$$

Proof. Choose $t_{R}>2 s_{0}+\frac{\theta_{0}}{2 \min \left\{\lambda\left(g_{s_{0}} x\right), \lambda\left(g_{-s_{0}} x\right\}\right.}$, where, again, $x$ is the point at which $R_{x}(\varepsilon)$ is centered. The proof of Lemma 2.27 gives us that for any $\langle\gamma, \eta\rangle \in R_{x}(\varepsilon)$, then $\gamma$ and $\langle\gamma, \eta\rangle$ turn at the same first cone point with the same angle, and that that cone point occurs in the interval $\left(0, t_{R}\right)$. Furthermore, by Lemma 2.27, for any $\langle\gamma, \eta\rangle \in R_{x}(\varepsilon), g_{t_{R}}\langle\gamma, \eta\rangle(t)=g_{t_{R}} \gamma(t)$ for $t \geq-t_{R}$. Therefore, $g_{t_{R}}\langle\gamma, \eta\rangle$ and $g_{t_{R}} \gamma$ agree at least up to the first cone point in negative time and for all cone points in positive time, and so $\lambda\left(g_{t+t_{R}}\langle\gamma, \eta\rangle\right)=\lambda\left(g_{t+t_{R}} \gamma\right)$ for all $t \geq 0$. A similar proof holds for $\eta$ by reversing the time direction.

Definition 2.33. Let $R_{x}(\varepsilon)$ be as in Definition 2.30. For $n \in \mathbb{N}$ such that $n>\frac{4}{\delta}$, we call $\mathcal{B}(n, \varepsilon)=$ $g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left(R_{x}(\varepsilon)\right)$ the $(n, \varepsilon)$-flow box centered at $x$.
Lemma 2.34. There exists $n_{0}=\max \left\{\frac{8}{\delta}, 5\right\}$, such that for all $n \geq n_{0}$, the Bowen bracket is defined on any pair of geodesics in $\mathcal{B}(n, \varepsilon)$ as in Definition 2.33.

Proof. Let $y, z \in \mathcal{B}(n, \varepsilon)$ then $y=g_{t_{y}}\left\langle\gamma_{y}, \eta_{y}\right\rangle$ and $z=g_{t_{z}}\left\langle\gamma_{z}, \eta_{z}\right\rangle$ for $\gamma_{y}, \gamma_{z} \in W^{u}(x, \varepsilon), \eta_{y}, \eta_{z} \in W^{s}(x, \varepsilon)$, and $t_{y}, t_{z} \in\left[-\frac{1}{n}, \frac{1}{n}\right]$. We claim that $[y, z]=g_{t_{z}}\left\langle\gamma_{y}, \eta_{z}\right\rangle$.

We have

$$
\begin{aligned}
d_{G S}\left(g_{t_{z}}\left\langle\gamma_{y}, \eta_{z}\right\rangle, y\right) & =d_{G S}\left(g_{t_{z}}\left\langle\gamma_{y}, \eta_{z}\right\rangle, g_{t_{y}}\left\langle\gamma_{y}, \eta_{y}\right\rangle\right) \\
& \leq \int_{-\infty}^{+\infty} d_{\tilde{S}}\left(\left\langle\gamma_{y}, \eta_{z}\right\rangle(t),\left\langle\gamma_{y}, \eta_{y}\right\rangle(t)\right) e^{-2\left|t-t_{z}\right|} d t+\left|t_{z}-t_{y}\right| \\
& =\int_{-\infty}^{0} d_{\tilde{S}}\left(\eta_{z}(t), \eta_{y}(t)\right) e^{-2\left|t-t_{z}\right|} d t+\left|t_{z}-t_{y}\right| \\
& \leq e^{\frac{2}{n}}\left(d_{G S}\left(\eta_{z}, x\right)+d_{G S}\left(\eta_{y}, x\right)\right)+\frac{2}{n} \\
& \leq e^{\frac{2}{n}} 2 \varepsilon+\frac{2}{n}<\delta
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
d_{G S}\left(g_{t_{z}}\left\langle\gamma_{y}, \eta_{z}\right\rangle, z\right) & =\int_{0}^{+\infty} d_{\tilde{S}}\left(\gamma_{y}(t), \gamma_{z}(t)\right) e^{-2\left|t-t_{z}\right|} d t \\
& \leq e^{2\left|t_{z}\right|}\left(d_{G S}\left(\gamma_{y}, x\right)+d_{G S}\left(\gamma_{z}, x\right)\right)=e^{\frac{2}{n}} 2 \varepsilon<\delta
\end{aligned}
$$

Thus, by the construction, we have that $g_{t_{z}}\left\langle\gamma_{y}, \eta_{z}\right\rangle \in W^{c s}(y, \delta) \cap W^{u}(z, \delta)$. Moreover, it is regular, so $[y, z]=g_{t_{z}}\left\langle\gamma_{y}, \eta_{z}\right\rangle$.

## 3. Dynamical Results

Definition 3.1. Given $\gamma \in G S$,

- The stable Bowen ball of length $n$ and radius $\varepsilon$ is

$$
B_{n}^{s}(\gamma, \epsilon):=\left\{\eta \in W^{s}(\gamma, \varepsilon) \mid d_{G S}\left(g_{t} \gamma, g_{t} \eta\right) \leq \epsilon \text { for }-n \leq t \leq 0\right\}
$$

- The center stable Bowen ball is

$$
B_{n}^{c s}(\gamma, \epsilon):=\left\{\eta \in W^{c s}(\gamma, \varepsilon) \mid d_{G S}\left(g_{t} \gamma, g_{t} \eta\right) \leq \epsilon \text { for }-n \leq t \leq 0\right\}
$$

- The unstable Bowen ball $B_{m}^{u}(\gamma, \epsilon)$ is defined similarly:

$$
B_{m}^{u}(\gamma, \varepsilon):=\left\{\eta \in W^{u}(\gamma, \varepsilon) \mid d_{G S}\left(g_{t} \gamma, g_{t} \eta\right) \leq \varepsilon \text { for } 0 \leq t \leq m\right\}
$$

except the shadowing occurs over the interval $0 \leq t \leq m$.

- The two-sided Bowen ball is

$$
B_{[-n, m]}(\gamma, \varepsilon):=\left\{\eta \in G S \mid d_{G S}\left(g_{t} \gamma, g_{t} \eta\right) \leq \varepsilon \text { for }-n \leq t \leq m\right\}
$$

We will also use the following notation $B_{t}(\gamma, \varepsilon):=B_{[0, t]}(\gamma, \varepsilon)$.
3.1. The global Bowen property. The Bowen property for a potential plays a key role in many of the techniques used for studying thermodynamic formalism of systems with some hyperbolicity.

Definition 3.2. A potential $\phi: G S \rightarrow \mathbb{R}$ has the global Bowen property if there is some $\varepsilon>0$ for which there exists a constant $K>0$ such that for all $x, y \in G S$ and $t>0$

$$
\sup \left\{\left|\int_{0}^{t} \phi\left(g_{r} x\right)-\phi\left(g_{r} y\right) d r\right|: d_{G S}\left(g_{r} x, g_{r} y\right) \leq \varepsilon \text { for } 0 \leq r \leq t\right\} \leq K
$$

In $\mathrm{CCE}^{+} 23$, we proved a non-uniform version of the Bowen property for any Hölder potential $\phi$, which we state explicitly below (Proposition 3.5). In this section we provide an argument which upgrades this nonuniform Bowen property to the global property for Hölder $\phi$ that are locally constant on a neighborhood of the singular set, using properties of $\lambda$. Using the $\lambda$-decomposition machinery we have developed already, the proof here is relatively short, and may be useful in other settings that use a $\lambda$-decomposition.

Before providing that proof, however, we note that it is possible to leverage the particularly nice geometry of $S$ to prove the global Bowen property without invoking the $\lambda$-decomposition machinery. The heart of the argument lies in studying the geometry of Bowen balls and how they interact with cone points. The figures below provide two illustrative examples of the possible geometry.


Figure 1. The boundary of $H$ intersects $x$ in two disjoint intervals.
Working in the universal cover, let $H$ be the convex hull of the balls of radius $\varepsilon$ around $\tilde{x}(0)$ and $\tilde{x}(t)$. Any geodesic $\tilde{y}$ in $B_{t}(\tilde{x}, \varepsilon)$ must traverse $H$. If $[\tilde{x}(0), \tilde{x}(t)]$ contains no cone points, for $r \in[T, t-T]$ where $T$ is as in Lemma 2.9, $g_{r} \tilde{x}$ is close to Sing and the fact that $\phi$ is constant near Sing can be used to prove the bound on $\left|\int_{0}^{t} \phi\left(g_{r} x\right)-\phi\left(g_{r} y\right) d r\right|$. Otherwise, the cone points on $[\tilde{x}(0), \tilde{x}(t)]$ have a strong impact on the geometry of $H$ - its boundary must pass through any cone points on $[\tilde{x}(T), \tilde{x}(t-T)]$. Figures 1 and 2 show two possible ways this can happen (cone points are marked by $\zeta_{i}$ ). In Figure 1, one can construct a singular geodesic $\tilde{\gamma}$ through $H$ and very near to both $\tilde{x}$ and $\tilde{y}$ using the cone-point-free 'line of sight' between the $\varepsilon$-balls around $\tilde{x}(0)$ and $\tilde{x}(t)$. Using again that $\phi$ is locally constant near Sing, the desired bound follows. In Figure 2, no such line of sight exists. But as $\tilde{y}$ traverses $H$ between $B_{\varepsilon}(\tilde{x}(0))$ and $\zeta_{1}$ and between $\zeta_{k}$ and $\tilde{x}(t)$


Figure 2. The boundary of $H$ intersects $x$ in $\left[\zeta_{1}, \zeta_{k}\right]$.
there are clear lines of sight and nearby singular geodesics. Between $\zeta_{1}$ and $\zeta_{k}, \tilde{x}$ and $\tilde{y}$ must overlap. Then Lemma 2.10 proves (after a small reparametrization) $g_{r} \tilde{x}$ and $g_{r} \tilde{y}$ are exponentially close, and the Hölder property of $\phi$ can be used to complete the result.

We now prove the global Bowen property using the $\lambda$-function.
Lemma 3.3. (compare to BCFT18, Proposition 3.4]) If $\lambda$ is lower semicontinuous, then for all $\delta>0$, there exist $c>0$ and $T>0$ such that if $\lambda\left(g_{t} x\right) \leq c$ for $t \in[-T, T]$, then $d_{G S}(x$, Sing $) \leq \delta$.
Proof. For all $c>0$, let $A(c)=\left\{x \in G S \mid \lambda\left(g_{t} x\right)>c\right.$ for some $\left.t \in\left[\frac{-1}{c}, \frac{1}{c}\right]\right\}$. Notice that by definition, $A\left(c_{2}\right) \subset A\left(c_{1}\right)$ if $0<c_{1}<c_{2}$.

We will show that $A(c)$ is open using lower semi-continuity of $\lambda$. Let $x \in A(c)$, and suppose that $\lambda\left(g_{t^{*}} x\right)>c$ for some $t^{*} \in\left[-c^{-1}, c^{-1}\right]$. Using lower semi-continuity, let $\epsilon_{1}>0$ be chosen small enough so that $\lambda(y)>c$ whenever $y \in B\left(g_{t^{*}} x, \epsilon_{1}\right)$. By continuity of the flow, let $\epsilon_{2}>0$ be chosen such that if $y \in B\left(x, \epsilon_{2}\right)$, then $g_{t^{*}} y \in B\left(g_{t^{*}} x, \epsilon_{1}\right)$. Then $B\left(x, \epsilon_{2}\right) \subset A(c)$, and so $A(c)$ is open.

By $\mathrm{CCE}^{+} 23$, Corollary 3.5], $\mathrm{Sing}^{c}=\bigcup_{c>0} A(c)$. Consider $K=\left\{x \in G S \mid d_{G S}(x\right.$, Sing $\left.) \geq \delta\right\} .\{A(c)\}_{c>0}$ is an open cover for the compact set $K$. Thus, there exists a finite subcover $\left\{A\left(c_{j}\right)\right\}_{j=1, \ldots, k}$ for $K$. Let $c=\min \left\{c_{1}, \ldots, c_{k}\right\}$. Since $A\left(c_{j}\right) \subset A(c)$ for $j=1, \ldots, k$, we find that $K \subset A(c)$ and the lemma holds.

Recall that $\mathcal{C}(c)$ is the set of orbit segments with $\lambda \geq c$ at their beginning and end (Definition 2.6).
Definition 3.4. Given a potential $\phi: G S \rightarrow \mathbb{R}$, we say that $\phi$ has the Bowen property on $\mathcal{C}(c)$ if there is some $\varepsilon>0$ for which there exists a constant $K>0$ such that

$$
\sup \left\{\left|\int_{0}^{t} \phi\left(g_{r} x\right)-\phi\left(g_{r} y\right) d r\right|:(x, t) \in \mathcal{C}(c) \text { and } d_{G S}\left(g_{r} y, g_{r} x\right) \leq \varepsilon \text { for } 0 \leq r \leq t\right\} \leq K
$$

Proposition 3.5. Any Hölder $\phi: G S \rightarrow \mathbb{R}$ has the Bowen property on $\mathcal{C}(c)$ for all $c>0$.
Proof. This is proved in $\S 6$ of $\left[\mathrm{CEE}^{+} 23\right]$. In that paper, we prove the Bowen property for a collection of orbit segments $\mathcal{G}(c)$ which is smaller than $\mathcal{C}(c)$; segments in $\mathcal{G}(c)$ have average value of $\lambda$ greater than $c$ over any initial and terminal segment, not just at times 0 and $t$. However, the proof only uses the values of $\lambda$ at $x$ and $g_{t} x$, and hence applies to $\mathcal{C}(c)$.

We now update this non-uniform result to the global Bowen property under the additional assumption that $\phi$ is locally constant on a neighborhood of Sing.
Proposition 3.6. If $\lambda$ is lower semicontinuous, $\phi$ is locally constant on $B(\operatorname{Sing}, \delta)$ for some $\delta>0$ and has the Bowen property on $\mathcal{C}(c)$ for all $c>0$, then $\phi$ has the global Bowen property.
Proof. Suppose that $\phi$ is constant on $B$ (Sing, $\delta$ ) for some $\delta>0$. Let $c$ and $T$ be as in Lemma 3.3 for $\frac{\delta}{2}$. Furthermore, assume $\phi$ has the Bowen property on $\mathcal{C}(c)$ for all $\epsilon<\epsilon_{0}$. Let $x \in G S$, let $\epsilon<\min \left\{\delta / 2, \epsilon_{0}\right\}$, and let $t>0$. Suppose that $d_{G S}\left(g_{s} y, g_{s} x\right) \leq \epsilon$ for $0 \leq s \leq t$.

We have four cases to consider, beginning with a trivial one.
Case 0: $t \leq 2 T$.
Trivially,

$$
\left|\int_{0}^{t} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right)\right| \leq 4 T\|\phi\|,
$$

where $\|\phi\|$ is the maximum value of $|\phi|$.
Henceforth we can assume $t>2 T$.
Case 1: $\lambda\left(g_{s} x\right)<c$ for all $s \in[0, t]$.
By Lemma 3.3, for $s \in[T, t-T]$, we have $g_{s} x \in B(\operatorname{Sing}, \delta / 2)$. Since $y \in B_{t}(x, \epsilon)$ and $\epsilon<\delta / 2, g_{s} y \in$ $B$ (Sing, $\delta$ ) for $s \in[T, t-T]$. Then as $\phi$ is locally constant on $B$ (Sing, $\delta$ ),

$$
\left|\int_{0}^{t} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right| \leq 4 T\|\phi\|
$$

as $\left|\int_{T}^{t-T} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right|=0$.
Henceforth, we can assume $\lambda$ takes on a value greater than or equal to $c$ at some point along the orbit segment $(x, t)$. Let $t_{1}=\min \left\{s \geq 0 \mid \lambda\left(g_{s} x\right) \geq c\right\}$ and $t_{2}=\max \left\{s \leq t \mid \lambda\left(g_{s} x\right) \geq c\right\}$. If $t_{1}=t_{2}$, then the simpler version of the estimates below work, so we assume $t_{2}>t_{1}$. Then $\left(g_{t_{1}} x, t_{2}-t_{1}\right) \in \mathcal{C}(c)$. Furthermore, for all $s \in\left[0, t_{1}\right] \cup\left[t_{2}, t\right]$, we have that $\lambda\left(g_{s} x\right) \leq c$. The proof is completed by applying our final two cases to the intervals $\left[0, t_{1}\right]$ and $\left[t_{2}, t\right]$ as appropriate.

Case 2: $t_{1} \leq 2 T$ and $t-t_{2} \leq 2 T$.
We present the situation where both of these inequalities hold. Applying the non-uniform Bowen property (Prop 3.5) for $\left(g_{t_{1}} x, t_{2}-t_{1}\right)$, we have

$$
\begin{aligned}
\left|\int_{0}^{t} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right| & \leq\left|\int_{0}^{t_{1}} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right|+\left|\int_{t_{1}}^{t_{2}} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right|+\left|\int_{t_{2}}^{t} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right| \\
& \leq 4 T\|\phi\|+K+4 T\|\phi\|
\end{aligned}
$$

where $K$ is given by the non-uniform Bowen property.
Case 3: $t_{1}>2 T$ or $t-t_{2}>2 T$.
We present the argument in a situation where both of these hold as, otherwise, we can use an estimate as in Case 0 for $\left|\int_{0}^{t_{1}} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right|$ or $\left|\int_{t_{2}}^{t} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right|$. By Lemma 3.3. for $s \in\left[T, t_{1}-T\right]$, we have $g_{s} x \in B$ (Sing, $\frac{\delta}{2}$ ), and similarly for $s \in\left[t_{2}+T, t-T\right]$. Then if $y \in B_{t}(x, \epsilon)$, because $\epsilon<\delta / 2$, it follows that $g_{s} y \in B(\operatorname{Sing}, \delta)$ for $s \in\left[T, t_{1}-T\right]$ and $s \in\left[t_{2}+T, t-T\right]$. Therefore, because $\phi$ is locally constant on $B$ (Sing, $\delta$ ), we have that

$$
\left|\int_{0}^{t_{1}} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right| \leq\left|\int_{0}^{T} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right|+0+\left|\int_{t_{1}-T}^{t_{1}} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right| \leq 4 T\|\phi\|
$$

Similarly, we can bound

$$
\left|\int_{t_{2}}^{t} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right| \leq 4 T\|\phi\| .
$$

Now, since $y \in B_{t}(x, \epsilon), g_{t_{1}} y \in B_{t_{2}-t_{1}}\left(g_{t_{1}} x, \epsilon\right)$. By Prop 3.5, there exists $K>0$ (independent of $\left.x, y, t_{1}, t_{2}\right)$ such that

$$
K \geq\left|\int_{0}^{t_{1}-t_{2}} \phi\left(g_{s}\left(g_{t_{1}} x\right)\right)-\phi\left(g_{s}\left(g_{t_{1}} y\right)\right) d s\right|=\left|\int_{t_{1}}^{t_{2}} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right|
$$

Combining this with our previous inequalities, we have

$$
\begin{aligned}
\left|\int_{0}^{t} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right| & \leq\left|\int_{0}^{t_{1}} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right|+\left|\int_{t_{1}}^{t_{2}} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right|+\left|\int_{t_{2}}^{t} \phi\left(g_{s} x\right)-\phi\left(g_{s} y\right) d s\right| \\
& \leq 4 T\|\phi\|+K+4 T\|\phi\|
\end{aligned}
$$

As our choice of $T$ and $K$ were independent of $x, t$, and $y$, this implies the Bowen property.

### 3.2. Upper and lower Gibbs properties.

3.2.1. Lower Gibbs. One of the results of CT16 shows that for any system satisfying the conditions of CT16. Theorem A], the unique equilibrium state has a type of lower Gibbs property. In [CEE ${ }^{+} 23$, we showed that in the setting of Theorems A and B with $\phi$ Hölder continuous and locally constant on Sing, $\left(G S, g_{t}, \mu, \phi\right)$ satisfy conditions sufficient to apply CT16. In particular, we show that the set of orbit segments $\mathcal{C}(c):=\left\{(x, t) \mid \lambda(x), \lambda\left(g_{t} x\right) \geq c\right\}$ satisfies the specification property in the proof of CCE ${ }^{+} 23$, $\S 4$ and §5]. Consequently, we have the following corollary.

Corollary 3.7. Let $\left(G S, \mu, g_{t}\right)$ be as in Theorems $A$ and $B$, and let $\phi$ be Hölder continuous and locally constant on Sing. Then $\mu$ has the following lower Gibbs property for a collection of orbit segments.

Let $\rho, \eta>0$. If $\left\{\left(x_{i}, t_{i}\right)\right\}$ is a collection of orbit segments in $G S \times[0, \infty)$ such that for all $i, \lambda\left(x_{i}\right), \lambda\left(g_{t_{i}} x_{i}\right) \geq$ $\eta$, then there exists a constant $Q_{2}(\rho, \eta)$ depending only on $\rho$ and $\eta$ such that for all $i$,

$$
\mu\left(B_{t_{i}}\left(x_{i}, \rho\right)\right) \geq Q_{2}(\rho, \eta) e^{-t_{i} P(\phi)+\Phi\left(x_{i}, t_{i}\right)}
$$

This gives us the following useful result.
Corollary 3.8. Let $\left(G S, \mu, g_{t}\right)$ be as in Theorems And B. Then $\mu\left(\lambda^{-1}(0)\right)=0$. In particular, $\mu(\operatorname{Sing})=0$.
Proof. Let $x \in G S$ be such that $\lambda(x)>0$. Then, because $\lambda$ is lower semi-continuous, there exists a ball $B(x, \rho)$ such that $\left.\lambda\right|_{B(x, \rho)}>0$. As the lower Gibbs property tells us that $\mu(B(x, \rho))>0$, this implies that $\mu$-a.e. $y \in G S$ visits $B(x, \rho)$ infinitely often in both forwards and backwards time. Thus, for almost every $y, \lambda(y) \neq 0$, as if $\lambda(y)=0$, then $\lambda\left(g_{t} y\right)=0$ either for all $t \geq 0$ or all $t \leq 0\left[\mathrm{CCE}^{+} 23\right.$, Proposition 3.4]. As Sing $\subset \lambda^{-1}(0)$, it follows that $\mu($ Sing $)=0$ as well.
3.2.2. Upper Gibbs. The upper Gibbs bound is improved from CT16 by our proof that $\phi$ has the Bowen property globally.

Proposition 3.9. Let $\left(G S, \mu, g_{t}\right)$ be as in Theorems $A$ and $B$. Then, $\mu$ has the upper Gibbs property. That is, for all $\rho>0$ sufficiently small, there exists $Q:=Q(\rho)>0$ such that

$$
\mu\left(B_{t}(x, \rho)\right) \leq Q e^{-t P(\phi)+\Phi(x, t)}
$$

Remark. We note that the upper bound on $\rho$ is less than half of the injectivity radius of $S$ (see $\mathrm{CCE}^{+} 23$, Lemmas 2.16 and 2.17]) and small enough that $\phi$ has the Bowen property.

Proof. Because in our setting we have specification at all scales [CCE ${ }^{+}$23, §5], applying [CT16, Proposition 4.21], for all sufficiently small $\rho>0$, there exists $Q^{\prime}>0$ such that

$$
\mu\left(B_{t}(x, \rho)\right) \leq Q^{\prime} e^{-t P(\phi)+\sup _{y \in B_{t}(x, \rho)} \Phi(y, t)}
$$

Since $\phi$ has the Bowen property at scale $\rho$ by Proposition 3.6, we have that

$$
\sup _{y \in B_{t}(x, \rho)} \Phi(y, t) \leq K+\Phi(x, t)
$$

Thus, it follows that

$$
\mu\left(B_{t}(x, \rho)\right) \leq Q^{\prime} e^{K} e^{-t P(\phi)+\Phi(x, t)}
$$

3.3. Local product structure for equilibrium states. A key tool in our proof of the Bernoulli property for $\left(G S, g_{t}, \mu\right)$ is local product structure of the equilibrium measure $\mu$. Roughly speaking, the stable, unstable, and flow directions provide a topological product structure near any regular geodesic. The equilibrium state $\mu$ has local product structure when it is absolutely continuous with respect to a product measure built using this topological local product structure (see Definition 3.26 for the precise definition).

As usual, the non-uniformly hyperbolic nature of our system only allows us to prove this structure near regular geodesics, with the constants involved depending on the level of regularity.

Our approach to the proof closely follows that of Cli23, where Climenhaga shows that the Gibbs property can be used to deduce local product structure. Our modifications to Climenhaga's scheme revolve around adjusting the argument to flows instead of maps (an easy change - replacing the stable leaves with centerstable leaves and using the invariance of $\mu$ under the flow) and (a much more significant change) accounting for the non-uniformity of our lower Gibbs bound (Corollary 3.7).

Sections 3.3.1 and 3.3 .2 provide, respectively, a technical construction of special partitions related to the Bowen balls, and preliminary results on Bowen balls and their relation to the topological local product structure. Section 3.3.3 is the heart of the matter, dedicated to proving Proposition 3.30 and Corollary 3.31 The local product structure of $\mu$ follows from these.
3.3.1. Partition Construction. Since the construction below might be of the independent interest, we formulate it in a considerably more general setting. Let $(X, d)$ be a compact metric space with the metric $d$, and let $\mathcal{F}=\left(f_{t}\right)_{t \in \mathbb{R}}: X \rightarrow X$ be a continuous flow. In this section, we construct an "almost adapted" partition of a compact subset of $X$ with a special property (see Proposition 3.12). We call this partition almost adapted as each element of the partition contains and is contained in a Bowen ball (see [Cli23, Definition 4.1] for the definition of adapted partitions). However, in our case the time parameter of the Bowen balls depends on the element of the partition.

We begin with a lemma that we will eventually use to construct our partitions.
Lemma 3.10. Suppose that a set $A$ has the following separation property: for each $x \in A$, there exists $t_{x}>0$ such that for any $x \neq y \in A$, $\max \left\{d\left(f_{t} x, f_{t} y\right) \mid t \in\left[0, \min \left\{t_{x}, t_{y}\right\}\right]\right\} \geq \varepsilon$. Then the collection $\left\{B_{t_{x}}\left(x, \frac{\varepsilon_{2}}{2}\right)\right\}_{x \in A}$ is pairwise disjoint.
Proof. Let $x \neq y \in A$. Without loss of generality, assume $t_{x} \leq t_{y}$. Towards a contradiction, suppose $z \in B_{t_{x}}\left(x, \frac{\varepsilon}{2}\right) \cap B_{t_{y}}\left(y, \frac{\varepsilon}{2}\right)$. Then, we have

$$
\begin{aligned}
\varepsilon \leq \max \left\{d\left(f_{t} x, f_{t} y\right) \mid t \in\left[0, t_{x}\right]\right\} & \leq \max \left\{d\left(f_{t} x, f_{t} z\right) \mid t \in\left[0, t_{x}\right]\right\}+\max \left\{d\left(f_{t} z, f_{t} y\right) \mid t \in\left[0, t_{x}\right]\right\} \\
& \leq \max \left\{d\left(f_{t} x, f_{t} z\right) \mid t \in\left[0, t_{x}\right]\right\}+\max \left\{d\left(f_{t} z, f_{t} y\right) \mid t \in\left[0, t_{y}\right]\right\} \\
& <\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,
\end{aligned}
$$

which is not possible. Therefore, $B_{t_{x}}\left(x, \frac{\varepsilon}{2}\right) \cap B_{t_{y}}\left(y, \frac{\varepsilon}{2}\right)=\emptyset$.
Definition 3.11. Let $A \subset X$. For $T, \varepsilon>0$ and $E \subset A$, we say that $E$ is a $(T, \varepsilon)$-separated subset of $A$ if $\max \left\{d\left(f_{t} x, f_{t} y\right) \mid t \in[0, T]\right\} \geq \varepsilon$ for all distinct $x, y \in E$.
Proposition 3.12. Consider a compact subset $K$ of $X$. Let $\lambda: X \rightarrow[0, \infty)$ be a non-negative function and assume $\lambda \circ f_{t}: K \rightarrow[0, \infty)$ is a continuous function for all $t \in \mathbb{R}$. Let $T_{0}, c>0$. Further, suppose there exists $\kappa>0$ such that for all $x \in K, \lambda\left(f_{T_{0}+n \kappa+r} x\right) \geq c$ for all $|r| \leq \kappa$ for infinitely many $n \in \mathbb{N}$. Then for all $\varepsilon>0, m \in \mathbb{N}$ there exists a finite partition $\xi_{m}$ of $K$ such that for each $A \in \xi_{m}$, there exist $x \in K$ and $t \geq T_{0}+m \kappa$ with $\lambda\left(f_{t+r} x\right) \geq c$ for $r \in[-\kappa, \kappa]$ and

$$
K \cap B_{t}\left(x, \frac{\varepsilon}{2}\right) \subset A \subset K \cap B_{t}(x, \varepsilon) .
$$

Further, for all $n \in \mathbb{N}$, we can choose $(x, t)$ so that $\lambda\left(f_{T_{0}+r_{i} \kappa} x\right) \geq c$ for $r_{1}<r_{2}<\cdots<r_{n}$ and $t \geq T_{0}+r_{n} \kappa$.
We call the partitions $\xi_{m}$ given by Proposition 3.12 almost adapted.
Remark. The essence of what we want from the points $x$ in this proposition is that their orbits continually return to $\{x \in X \mid \lambda(x) \geq c\}$ and hence experience the hyperbolicity of the flow. The precise conditions on $x$ in this proposition (in particular, remaining in $\{x \in X \mid \lambda(x) \geq c\}$ over the interval $[t-\kappa, t+\kappa]$ ) are imposed to make later arguments slightly simpler (in particular, allowing us to remove some cases from a future argument). However, as we note below in Lemma 3.14, while the condition we impose on $x$ may seem complicated, it is not too difficult to check.
Proof. Let $T=T_{0}+m \kappa$. First, we define $K_{0}:=\left\{x \in K \mid \lambda\left(f_{T+r} x\right) \geq c\right.$ for $\left.|r| \leq \kappa\right\}$. We note that $K_{0}$ is compact using the facts that $\lambda \circ f_{T}$ is continuous and $K$ is compact. Additionally, for each $x \in K$ and $i \in \mathbb{N}$, let $\Psi_{i}(x)$ be the number of $p \in \mathbb{N}$ such that both $T_{0}+p \kappa \leq T+i \kappa$ and $\lambda\left(f_{T_{0}+p \kappa} x\right) \geq c$. Then, defining

$$
F_{0}:=\left\{x \in K_{0} \mid \Psi_{0}(x) \geq n\right\},
$$

we have that $F_{0}$ is compact, as the set $\left\{x \in K \mid \Psi_{0}(x) \geq n\right\}$ is compact (it is a finite union of compact sets). Let $A_{0}$ be a maximal $(T, \varepsilon)$-separated subset of $F_{0}$ that exists because $F_{0}$ is compact. Then, for all $i \in \mathbb{N}$,
we define $K_{i}$ and $F_{i}$ inductively in the following way. We set

$$
K_{i}:=\left\{x \in K \mid \lambda\left(f_{T+i \kappa+r} x\right) \geq c \text { for }|r| \leq \kappa\right\} \backslash \bigcup_{j=0}^{i-1} \bigcup_{x \in A_{j}}\left(K \cap B_{T+j \kappa}(x, \varepsilon)\right)
$$

let

$$
F_{i}:=\left\{x \in K_{i} \mid \Psi_{i}(x) \geq n\right\}
$$

and let $A_{i}$ be a maximal $(T+i \kappa, \varepsilon)$-separated subset of $F_{i}$. We note that $F_{i}$ is compact for all $i \in \mathbb{N}$ so $A_{i}$ exists. Observe from this construction that $\left\{B_{T+i \kappa}(x, \varepsilon) \mid 0 \leq i \leq N, x \in A_{i}\right\}$ is an open cover of $\bigcup_{i=0}^{N} F_{i}$, as otherwise, there would be some $0 \leq i \leq N$ and $y \in F_{i}$ for which $A_{i} \cup\{y\}$ is $(T+i \kappa, \varepsilon)$-separated. Since for all $x \in K, \lambda\left(f_{T+i \kappa} x\right) \geq c$ for infinitely many $i \in \mathbb{N}$, there exists some $N$ for which $g_{N}(x) \geq n$ and so either $x \in F_{N}$ or $x \in \bigcup_{i=0}^{N-1} \bigcup_{x \in A_{i}}\left(K \cap B_{T+i \kappa}(x, \varepsilon)\right)$. Thus, $\left\{B_{T+n \kappa}(x, \varepsilon) \mid n \in \mathbb{N}, x \in A_{n}\right\}$ is an open cover of $K$.

Since $K$ is compact, we can choose a minimal, finite subcover $\left\{B_{T+n_{i} \kappa}\left(x_{i}, \varepsilon\right)\right\}_{i=1}^{k}$. The minimality guarantees that $x_{i} \neq x_{j}$ for $i \neq j$.

Claim 3.13. $\max \left\{d\left(f_{t} x_{i}, f_{t} x_{j}\right) \mid t \in\left[0, T+\min \left\{n_{i}, n_{j}\right\} \kappa\right]\right\} \geq \varepsilon$ for $i \neq j$.
Proof. Without loss of generality, we can assume $n_{i} \leq n_{j}$. If $n_{i}=n_{j}$, the claim follows from the fact that $A_{n_{i}}$ is $\left(T+n_{i} \kappa, \varepsilon\right)$-separated. If $n_{i}<n_{j}$, then $x_{j} \notin B_{T+n_{i} \kappa}\left(x_{i}, \varepsilon\right)$ by construction (as $x_{j} \in K_{n_{j}}$ ), completing our proof.

By Claim 3.13 and Lemma 3.10, $\left\{B_{T+n_{i} \kappa}\left(x_{i}, \frac{\varepsilon}{2}\right)\right\}_{i=1}^{k}$ are pairwise disjoint. Finally, we construct the partition $\xi_{m}=\left\{P_{1}, \ldots, P_{k}\right\}$ in the following way by induction. Let $P_{1}=B_{T+n_{1} \kappa}\left(x_{1}, \varepsilon\right) \cap K$. Then, for $i=2, \ldots, k$, we have $P_{i}=K \cap\left(B_{T+n_{i} \kappa}\left(x_{i}, \varepsilon\right) \backslash \bigcup_{j=1}^{i-1} P_{j}\right)$.

Lemma 3.14. Let $\kappa>0, x \in X$, and suppose there exists a sequence $t_{k} \rightarrow \infty$ for which $\lambda\left(f_{t_{k}+s} z\right) \geq c$ for all $|s| \leq 2 \kappa$. Then for all $T_{0}>0, \lambda\left(f_{T_{0}+n \kappa+s} x\right) \geq c$ for all $|s| \leq \kappa$ and infinitely many $n$.
Proof. Let $T_{0}>0$. Then for any $t_{k} \geq T_{0}$, write $t_{k}=T_{0}+n_{k} \kappa+r$ for some $0 \leq r<\kappa$. Then, observe that for all $|s| \leq \kappa$, we have that $T_{0}+n_{k} \kappa+s=t_{k}+s-r$ and $|s-r| \leq 2 \kappa$. By assumption on $t_{k}$, we have completed our proof.

Corollary 3.15. Assume we are in the setting of Proposition 3.12. If $K$ has an additional property that for all $x \in K$, $\lim _{t \rightarrow \infty} \operatorname{diam}\left(B_{t}(x, \varepsilon) \cap K\right)=0$, then there exists an almost adapted partition $\xi$ of $K$ with arbitrarily small diameter.

Proof. Let $\alpha>0$ be arbitrary. By Dini's Theorem, $\operatorname{diam}\left(B_{t}(x, \varepsilon) \cap K\right) \rightarrow 0$ as $t \rightarrow \infty$ uniformly, because $x \mapsto \operatorname{diam} B_{t}(x, \varepsilon)$ is continuous. Then, choose $m \in \mathbb{N}$ in Proposition 3.12 large enough so that $\sup _{x \in K} \operatorname{diam} B_{t}(x, \varepsilon) \leq \alpha$ for all $t \geq T_{0}+m \kappa$.
$x \in K$
3.3.2. Interaction of Bowen balls and Bowen brackets. Throughout this section we use the following notations. Let $z_{0} \in G S$ be such that $\lambda\left(z_{0}\right) \geq \frac{3}{4}\|\lambda\|_{\infty}$, and, using lower semicontinuity of $\lambda$, let $\kappa>0$ be chosen so that $\left.\lambda\right|_{B\left(z_{0}, 3 \kappa\right)} \geq \frac{2}{3} \lambda\left(z_{0}\right)$. Consider a parametrized geodesic $x_{0}$ and $\delta$ that satisfy Lemma 2.27 , $\varepsilon<\frac{1}{4} \min \left\{s_{0}, \delta\right\}$, and $n_{0}=\max \left\{\frac{8}{\delta}, 5, \frac{1}{s_{0}}, \frac{4}{\kappa}, \frac{1}{\varepsilon}\right\}$. Let $R(\varepsilon)$ be an su-rectangle centered at $x_{0}$ (see Definiton 2.30), and $\mathcal{B}\left(n_{0}, \varepsilon\right)$ be the corresponding $\left(n_{0}, \varepsilon\right)$-flow box (see Definition 2.33).

The following notation will prove useful throughout the remainder of the paper.
Definition 3.16. Given a flow box $\mathcal{B}\left(n_{0}, \varepsilon\right)$ and a point $x \in \mathcal{B}\left(n_{0}, \varepsilon\right)$, let

$$
V_{x}^{c s}=W^{c s}(x, \delta) \cap \mathcal{B}(n, \varepsilon) \quad \text { and } \quad V_{x}^{u}=W^{u}(x, \delta) \cap \mathcal{B}(n, \varepsilon)
$$

be the local center stable and unstable leaves induced on $\mathcal{B}\left(n_{0}, \varepsilon\right)$.
Lemma 3.17. For all $0<a \leq 2 \varepsilon$ and $n \geq n_{0}$,

$$
\frac{\mu\left(g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle^{s u}\right)}{\mu\left(g_{[-a, a]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle^{s u}\right)}=\frac{1}{n a}
$$

Proof. We prove the more general statement, where $A \subset G S$ is such that for all $t \neq s$ with $|t|,|s|$ sufficiently small, $g_{t} A \cap g_{s} A=\emptyset$. By flow invariance of $\mu$, for all small $t_{1}, t_{2}, s$, we have $\mu\left(g_{\left[t_{1}, t_{2}\right]} A\right)=\mu\left(g_{\left[s+t_{1}, s+t_{2}\right]} A\right)$. For $0<a \leq 2 \varepsilon$ and $n \geq n_{0}$, we have the necessary disjointness property above for $\left|t_{i}\right| \leq a, \frac{1}{n}$, as $a$ and $\frac{1}{n}$ are less than one quarter the injectivity radius of $S$. Thus, consider $g_{\left[\frac{-a}{n}, \frac{a}{n}\right]} A$, and observe that

$$
\frac{1}{n} \mu\left(g_{[-a, a]} A\right)=a \mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]} A\right)=\mu\left(g_{\left[\frac{-a}{n}, \frac{a}{n}\right]} A\right) .
$$

Lemma 3.18. Suppose $w, z \in g_{\tau} R(\varepsilon)$ for some $\tau \in\left[-\frac{1}{n_{0}}, \frac{1}{n_{0}}\right]$. Then, for all $n, m \geq 0$,

$$
g_{\left[-\frac{1}{n_{0}}, \frac{1}{n_{0}}\right]}\left\langle B_{m}^{u}(w, \varepsilon), B_{n}^{s}(z, \varepsilon)\right\rangle \subset B_{[-n, m]}(\langle w, z\rangle, 4 \varepsilon) .
$$

Proof. Let $w^{\prime} \in B_{m}^{u}(w, \varepsilon), z^{\prime} \in B_{n}^{s}(z, \varepsilon)$. Then, write $\xi:=\left\langle w^{\prime}, z\right\rangle, \eta:=\left\langle w^{\prime}, z^{\prime}\right\rangle$, and $\zeta:=\langle w, z\rangle$. Then, for $-n \leq k \leq 0$, we have

$$
\begin{aligned}
d_{G S}\left(g_{k} \eta, g_{k} \xi\right) & =\int_{-\infty}^{+\infty} e^{-2|t|} d_{\tilde{S}}\left(g_{k} \tilde{\xi}(t), g_{k} \tilde{\eta}(t)\right) d t \\
& =\int_{-\infty}^{+\infty} e^{-2|t|} d_{\tilde{S}}(\tilde{\xi}(t+k), \tilde{\eta}(t+k)) d t \\
& =\int_{-\infty}^{-(k+\tau)} e^{-2|t|} d_{\tilde{S}}(\tilde{\xi}(t+k), \tilde{\eta}(t+k)) d t+\int_{-(k+\tau)}^{+\infty} e^{-2|t|} d_{G \tilde{S}}(\tilde{\xi}(t+k), \tilde{\eta}(t+k)) d t \\
& =\int_{-\infty}^{-(k+\tau)} e^{-2|t|} d_{\tilde{S}}\left(\tilde{z}(t+k), \tilde{z^{\prime}}(t+k)\right) d t \\
& \leq \int_{-\infty}^{+\infty} e^{-2|t|} d_{\tilde{S}}\left(\tilde{z}(t+k), \tilde{z^{\prime}}(t+k)\right) d t \\
& \leq d_{G \tilde{S}}\left(g_{k} \tilde{z}, g_{k} \tilde{z}^{\prime}\right) \\
& \leq \varepsilon
\end{aligned}
$$

Furthermore, as $\eta \in W^{s}(\xi, \varepsilon)$, it follows that $d_{G S}\left(g_{k} \eta, g_{k} \xi\right) \leq \varepsilon$ for all $k \geq 0$ as well.
Now, using a similar argument, we see that for $0 \leq k \leq m$ (and indeed, for all $k \leq 0$ ), we have

$$
\begin{aligned}
d_{G S}\left(g_{k} \zeta, g_{k} \xi\right) & =\int_{-\infty}^{+\infty} e^{-2|t|} d_{\tilde{S}}\left(g_{k} \tilde{\xi}(t), g_{k} \tilde{\zeta}(t)\right) d t \\
& =\int_{-\infty}^{+\infty} e^{-2|t|} d_{\tilde{S}}(\tilde{\xi}(t+k), \tilde{\zeta}(t+k)) d t \\
& =\int_{-\infty}^{-(k+\tau)} e^{-2|t|} d_{\tilde{S}}(\tilde{\xi}(t+k), \tilde{\zeta}(t+k)) d t+\int_{-(k+\tau)}^{+\infty} e^{-2|t|} d_{G \tilde{S}}(\tilde{\xi}(t+k), \tilde{\zeta}(t+k)) d t \\
& =\int_{-(k+\tau)}^{+\infty} e^{-2|t|} d_{\tilde{S}}\left(\tilde{w^{\prime}}(t+k), \tilde{w}(t+k)\right) d t \\
& \leq \int_{-\infty}^{+\infty} e^{-2|t|} d_{\tilde{S}}\left(\tilde{w}^{\prime}(t+k), \tilde{w}(t+k)\right) d t \\
& \leq d_{G \tilde{S}}\left(g_{k} \tilde{w}^{\prime}, g_{k} \tilde{w}\right) \\
& \leq \varepsilon
\end{aligned}
$$

Therefore, we see that for $-n \leq k \leq m$, we have

$$
d_{G S}\left(g_{k} \eta, g_{k} \zeta\right) \leq d_{G S}\left(g_{k} \eta, g_{k} \xi\right)+d_{G S}\left(g_{k} \xi, g_{k} \zeta\right) \leq 2 \varepsilon
$$

Consequently, for $s \in\left[-\frac{1}{n_{0}}, \frac{1}{n_{0}}\right]$ and $-n \leq k \leq m$, we have

$$
d_{G S}\left(g_{s} g_{k} \eta, g_{k} \zeta\right) \leq d_{G S}\left(g_{s} g_{k} \eta, g_{k} \eta\right)+d_{G S}\left(g_{k} \eta, g_{k} \zeta\right) \leq|s|+2 \varepsilon \leq 4 \varepsilon
$$

Lemma 3.19. Suppose $w, z \in g_{\tau} R(\varepsilon)$ for some $\tau \in\left[-\frac{1}{n_{0}}, \frac{1}{n_{0}}\right]$. Then for all $n, m \geq 0$, we have

$$
B_{[-n, m]}(\langle w, z\rangle, \varepsilon) \subset g_{\left[-\varepsilon+\frac{1}{n_{0}}, \varepsilon+\frac{1}{n_{0}}\right]}\left\langle B_{m}^{u}(w, 3 \varepsilon), B_{n}^{s}(z, 3 \varepsilon)\right\rangle
$$

Proof. First, for ease of notation, set $\xi:=\langle w, z\rangle$. Now let $x \in B_{[-n, m]}(\xi, \varepsilon)$. We will show that there exists $s \in\left[-\varepsilon+\frac{1}{n_{0}}, \varepsilon+\frac{1}{n_{0}}\right]$ such that $g_{s} x \in R(\varepsilon)$. By our choice of $\varepsilon \leq \frac{\delta}{4}, x$ shares a geodesic segment around time $-\tau$ with $\xi$ (see the proof of Lemma 2.27). Consequently, there exists $s \in \mathbb{R}$ such that $x(s)=\xi(-\tau)$, and by assumption on $x$, we know that $s \in[-\tau-\varepsilon,-\tau+\varepsilon]$.

With that $s$ fixed, we will show that $g_{-s} x \in\left\langle B_{m}^{u}(w, 3 \varepsilon), B_{n}^{s}(z, 3 \varepsilon)\right\rangle$. A key observation will be that for all $k \in \mathbb{N}$, we have

$$
\begin{aligned}
d_{G S}\left(g_{k} x, g_{k} \xi\right) & =\int_{-\infty}^{+\infty} e^{-2|t|} d_{\tilde{S}}(\tilde{x}(t+k), \tilde{\xi}(t+k)) d t \\
& =\int_{-\infty}^{-k} e^{-2|t|} d_{\tilde{S}}(\tilde{x}(t+k), \tilde{z}(t+k)) d t+\int_{-k}^{+\infty} e^{-2|t|} d_{\tilde{S}}(\tilde{x}(t+k), \tilde{w}(t+k)) d t
\end{aligned}
$$

Now, set $\eta:=\left\langle g_{-s} x, w\right\rangle$ and $\zeta:=\left\langle z, g_{-s} x\right\rangle$. Let $0 \leq k \leq m$. Then observe

$$
\begin{aligned}
d_{G S}\left(g_{k} \eta, g_{k} w\right) & =\int_{-\infty}^{+\infty} e^{-2|t|} d_{\tilde{S}}(\tilde{\eta}(t+k), \tilde{w}(t+k)) d t \\
& =\int_{-k}^{+\infty} e^{-2|t|} d_{\tilde{S}}\left(g_{-s} \tilde{x}(t+k), \tilde{w}(t+k)\right) d t \\
& \leq \int_{-k}^{+\infty} e^{-2|t|} s d t+\int_{-k}^{+\infty} e^{-2|t|} d_{\tilde{S}}(\tilde{x}(t+k), \tilde{w}(t+k)) d t \\
& \leq s+d_{G S}\left(g_{k} x, g_{k} \xi\right) \\
& \leq 3 \varepsilon
\end{aligned}
$$

A similar argument holds for $-n \leq k \leq 0$.
We can use our previous lemmas to provide estimates using the Gibbs properties.
Lemma 3.20. Let $x \in \mathcal{B}\left(n_{0}, \varepsilon\right)$ and $w \in V_{x}^{u}$. There exists $Q_{1}:=Q_{1}(\varepsilon)$ such that for all $n \geq n_{0}$ and for all $m>0$,

$$
\mu\left(g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle B_{m}^{u}(w, \varepsilon), B_{n}^{s}(x, \varepsilon)\right\rangle\right) \leq \frac{n_{0}}{n} Q_{1} e^{-(n+m) P(\phi)+\int_{-n}^{m} \phi\left(g_{s} w\right) d s}
$$

Proof. By Lemma 3.17 , it suffices to estimate $\mu\left(g_{\left[-\frac{1}{n_{0}}, \frac{1}{n_{0}}\right.}\left\langle B_{m}^{u}(w, \varepsilon), B_{n}^{s}(x, \varepsilon)\right\rangle\right)$. Then, appealing to Lemma 3.18 and the upper Gibbs property (Proposition 3.9),

$$
\begin{aligned}
\mu\left(g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle B_{m}^{u}(w, \varepsilon), B_{n}^{s}(x, \varepsilon)\right\rangle\right) & =\frac{n_{0}}{n} \mu\left(g_{\left[-\frac{1}{n_{0}}, \frac{1}{n_{0}}\right]}\left\langle B_{m}^{u}(w, \varepsilon), B_{n}^{s}(x, \varepsilon)\right\rangle\right) \\
& \leq \frac{n_{0}}{n} \mu\left(B_{[-n, m]}(\langle w, x\rangle, 4 \varepsilon)\right) \\
& \leq \frac{n_{0}}{n} Q_{1} e^{-(n+m) P(\phi)+\int_{-n}^{m} \phi\left(g_{s}\langle w, x\rangle\right) d s} .
\end{aligned}
$$

While the estimate on the upper bound was straightforward, the lower bound is slightly more complicated, due to the fact that the lower Gibbs property of $\mu$ is non-uniform.

Lemma 3.21. Let $x \in \mathcal{B}\left(n_{0}, \varepsilon\right)$ and $w \in V_{x}^{u}$. Then for all $n \geq n_{0}$ and $m \geq 0$, there exists $Q_{2}:=$ $Q_{2}\left(\varepsilon, \lambda\left(g_{-n} w\right), \lambda\left(g_{m} w\right)\right)$ such that

$$
\mu\left(g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle B_{m}^{u}(w, \varepsilon), B_{n}^{s}(x, \varepsilon)\right\rangle\right) \geq \frac{n_{0}}{n\left(n_{0} \varepsilon+1\right)} Q_{2}\left(\varepsilon, \lambda\left(g_{-n} w\right), \lambda\left(g_{m} w\right)\right) e^{-(n+m) P(\phi)+\int_{-n}^{m} \phi\left(g_{s} w\right) d s}
$$

Proof. We again use Lemma 3.17, together with Lemma 3.19 and Corollary 3.7 to obtain:

$$
\begin{aligned}
\mu\left(g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle B_{m}^{u}(w, \varepsilon), B_{n}^{s}(x, \varepsilon)\right\rangle\right) & =\frac{n_{0}}{n\left(n_{0} \varepsilon+1\right)} \mu\left(g_{\left[-\left(\varepsilon+\frac{1}{n_{0}}\right), \varepsilon+\frac{1}{\left.n_{0}\right]}\right.}\left\langle B_{m}^{u}(w, \varepsilon), B_{n}^{s}(x, \varepsilon)\right\rangle\right) \\
& \geq \frac{n_{0}}{n\left(n_{0} \varepsilon+1\right)} \mu\left(B_{[-n, m]}\left(\langle w, x\rangle, \frac{\varepsilon}{3}\right)\right) \\
& \geq \frac{n_{0}}{n\left(n_{0} \varepsilon+1\right)} Q_{2}\left(\varepsilon, \lambda\left(g_{-n} w\right), \lambda\left(g_{m} w\right)\right) e^{-(n+m) P(\phi)+\int_{-n}^{m} \phi\left(g_{s} w\right) d s}
\end{aligned}
$$

We now prove a few more helpful lemmas.
Lemma 3.22. Given $x, y \in \mathcal{B}\left(n_{0}, \varepsilon\right)$, the function $\pi_{x, y}: V_{x}^{u} \rightarrow V_{y}^{u}$ defined by $\pi_{x, y}(z):=[z, y]$ is Lipschitz continuous with constant $e^{\frac{4}{n_{0}}}$. Furthermore, there exists $|\rho| \leq \frac{2}{n_{0}}$ such that $\pi_{x, y}(z):=\left\langle g_{\rho} z, y\right\rangle$ for all $z \in V_{x}^{u}$, and $\pi_{y, x}(z):=\left\langle g_{-\rho} z, x\right\rangle$.

Proof. We begin by showing the second part of this lemma. First, we know that there exists $r, s \in$ $\left[-\frac{1}{n_{0}}, \frac{1}{n_{0}}\right]$ such that $g_{r} x, g_{s} y \in R(\varepsilon)$. Consequently, $\left\langle g_{r} x, g_{s} y\right\rangle$ is defined and we have that $\left\langle g_{r} x, g_{s} y\right\rangle( \pm \infty)=$ $[x, y]( \pm \infty)$. As $[x, y]$ consists of only one geodesic (because it contains a regular geodesic), it follows that there exists $\tau$ such that $g_{\tau}\left\langle g_{r} x, g_{s} y\right\rangle=[x, y]$. Because we know that $[x, y] \in W^{u}\left(y, d_{G S}(x, y)\right)$ and $\left\langle g_{r} x, g_{s} y\right\rangle \in W^{u}\left(g_{s} y, d_{G S}\left(g_{s} y,\left\langle g_{r} x, g_{s} y\right\rangle\right)\right.$, by Corollary 2.18, it follows that $\tau=-s$, and so we have that $[x, y]=g_{-s}\left\langle g_{r} x, g_{s} y\right\rangle$. Now, it follows that $[x, y]=\left\langle g_{r-s} x, y\right\rangle$ by Corollary 2.18, and the fact that $d_{G S}\left(\left\langle g_{r-s} x, y\right\rangle, y\right) \leq d_{G S}\left(g_{r-s} x, y\right)$. Thus, taking $\rho=r-s$, we have $|\rho| \leq \frac{2}{n_{0}}$, and complete the proof of the second part of the lemma.

For the first, observe that

$$
\begin{aligned}
d_{G S}([w, y],[z, y]) & =d_{G S}\left(\left\langle g_{\rho} w, y\right\rangle,\left\langle g_{\rho} z, y\right\rangle\right) \\
& \leq d_{G S}\left(g_{\rho} w, g_{\rho} z\right) \\
& \leq e^{2|\rho|} d_{G S}(w, z)
\end{aligned}
$$

The first inequality follows from the description of how $\langle-,-\rangle$ works within $\mathcal{B}\left(n_{0}, \varepsilon\right)$ given by Lemma 2.27 Specifically, $\left\langle g_{\rho} w, y\right\rangle(r)=\left\langle g_{\rho} w, y\right\rangle(r)=y(r)$ for all $r \leq 0$ but for $r \geq 0,\left\langle g_{\rho} w, y\right\rangle(r)=g_{\rho} w(r)$ and $\left\langle g_{\rho} z, y\right\rangle(r)=g_{\rho} z(r)$.

Corollary 3.23. Given $x \in \mathcal{B}\left(n_{0}, \varepsilon\right)$, the function $[\cdot, x]: \mathcal{B}\left(n_{0}, \varepsilon\right) \rightarrow V_{x}^{u}$ is Lipschitz continuous, with constant $e^{\frac{2}{n_{0}}}$.

Proof. Let $y, z \in \mathcal{B}\left(n_{0}, \varepsilon\right)$. Then, since $[y, x]=[[y, z], x]$ and $d_{G S}(z,[y, z]) \leq d_{G S}(y, z)$, we have that

$$
d_{G S}([y, x],[z, x])=d_{G S}\left(\pi_{z, x}([y, z]), \pi_{z, x}(z)\right) \leq e^{\frac{2}{n_{0}}} d_{G S}([y, z], z) \leq e^{\frac{4}{n_{0}}} d_{G S}(y, z)
$$

Lemma 3.24. Let $\gamma \in \operatorname{Reg}$ and fix $s \in \mathbb{R}$. Then for all $t>|s|$ we have

$$
B_{t+|s|}^{u}\left(g_{s} \gamma, \varepsilon\right) \subset g_{s} B_{t}^{u}(\gamma, \varepsilon) \subset B_{t-|s|}^{u}\left(g_{s} \gamma, \varepsilon\right)
$$

Proof. First,

$$
g_{s} B_{t}^{u}(\gamma, \varepsilon)=\left\{g_{s} \eta \in W^{u}\left(g_{s} \gamma, \varepsilon\right) \mid d_{G S}\left(g_{t-s} g_{s} \gamma, g_{t-s} g_{s} \eta\right)<\varepsilon\right\}
$$

Since $t-|s| \leq t-s \leq t+|s|$, and since the geodesics in question are on the same unstable leaf,

$$
d_{G S}\left(g_{t-|s|} g_{s} \gamma, g_{t-|s|} g_{s} \eta\right) \leq d_{G S}\left(g_{t-s} g_{s} \gamma, g_{t-s} g_{s} \eta\right) \leq d_{G S}\left(g_{t+|s|} g_{s} \gamma, g_{t+|s|} g_{s} \eta\right)
$$

Therefore, if $g_{s} \eta \in B_{t+|s|}^{u}\left(g_{s} \gamma, \varepsilon\right), d_{G S}\left(g_{t-s} g_{s} \gamma, g_{t-s} g_{s} \eta\right)<\varepsilon$ and so $g_{s} \eta \in g_{s} B_{t}^{u}(\gamma, \varepsilon)$. Similarly, if $g_{s} \eta \in$ $g_{s} B_{t}^{u}(\gamma, \varepsilon)$, then $d_{G S}\left(g_{t-|s|} g_{s} \gamma, g_{t-|s|} g_{s} \eta\right)<\varepsilon$ and $g_{s} \eta \in B_{t-|s|}^{u}\left(g_{s} \gamma, \varepsilon\right)$.

Lemma 3.25. For $\gamma, \eta \in R(\varepsilon)$, for all $0<\varepsilon^{\prime} \leq \varepsilon$, we have

$$
B_{t}^{u}\left(\langle\gamma, \eta\rangle, \varepsilon^{\prime}\right)=\left\langle B_{t}^{u}\left(\gamma, \varepsilon^{\prime}\right), \eta\right\rangle
$$

Proof. First, let $\xi \in B_{t}^{u}\left(\langle\gamma, \eta\rangle, \varepsilon^{\prime}\right)$. Then, because $\xi \in R(\varepsilon)$, observe that

$$
\xi=\langle\xi, \eta\rangle=\langle\langle\xi, \gamma\rangle, \eta\rangle .
$$

Thus, we need to show that $\langle\xi, \gamma\rangle \in B_{t}^{u}\left(\gamma, \varepsilon^{\prime}\right)$. First, note that for $r \leq 0$, we have

$$
\langle\xi, \gamma\rangle(r)=\gamma(r)
$$

while for $r \geq 0$, we have

$$
\langle\xi, \gamma\rangle(r)=\xi(r),
$$

as well as

$$
\langle\gamma, \eta\rangle(r)=\gamma(r)
$$

Then, for $0 \leq r \leq t$, we have

$$
\begin{aligned}
d_{G S}\left(g_{r}\langle\xi, \gamma\rangle, g_{r} \gamma\right) & =\int_{-\infty}^{+\infty} e^{-2|s|} d_{S}(\langle\xi, \gamma\rangle(s+r), \gamma(s+r)) d s \\
& =\int_{-r}^{+\infty} e^{-2|s|} d_{S}(\xi(s+r), \gamma(s+r)) d s \\
& =\int_{-r}^{+\infty} e^{-2|s|} d_{S}(\xi(s+r),\langle\gamma, \eta\rangle(s+r)) d s \\
& \leq d_{G S}\left(g_{r} \xi, g_{r}\langle\gamma, \eta\rangle\right) \\
& <\varepsilon^{\prime} .
\end{aligned}
$$

This shows that $B_{t}^{u}\left(\langle\gamma, \eta\rangle, \varepsilon^{\prime}\right) \subset\left\langle B_{t}^{u}\left(\gamma, \varepsilon^{\prime}\right), \eta\right\rangle$. For the reverse inclusion, let $\xi \in B_{t}^{u}\left(\gamma, \varepsilon^{\prime}\right)$. Then $\langle\xi, \eta\rangle$ is in the unstable set of of $\eta$, and therefore of $\langle\gamma, \eta\rangle$ as desired. Furthermore, we have that for $0 \leq r \leq t$,

$$
d_{G S}\left(g_{r}\langle\xi, \eta\rangle, g_{r}\langle\gamma, \eta\rangle\right) \leq d_{G S}\left(g_{r} \xi, g_{r} \gamma\right) \leq \varepsilon^{\prime}
$$

This completes our proof.
3.3.3. Proof of the local product structure. In this section, we continue the same choices of $R(\varepsilon)$ and $\mathcal{B}\left(n_{0}, \varepsilon\right)$. The local product structure we want from $\mu$ is the following.

Definition 3.26. A measure $\mu$ has local product structure on $\mathcal{B}(n, \varepsilon)$ if there exist $x \in \mathcal{B}(n, \varepsilon)$ and measures $\nu^{c s, u}$ on $V_{x}^{c s, u}$ such that $\left.\mu\right|_{\mathcal{B}(n, \varepsilon)} \ll \nu^{u} \otimes \nu^{c s}$ where $\nu^{u} \otimes \nu^{c s}$ is the measure on $\mathcal{B}(n, \varepsilon)$ given by pushing forward the product measure $\nu^{u} \times \nu^{c s}$ under $[-,-]: V_{x}^{u} \times V_{x}^{c s} \rightarrow \mathcal{B}(n, \varepsilon)$.

The measures $\nu^{c s, u}$ will be conditional measures induced by $\mu$ on $V_{x}^{c s, u}$, and proving this can be reduced to understanding how conditional measures on the local unstable leaves behave under the holonomy map, which maps from one unstable leaf to another along the center-stable leaves. The key result of this section is Proposition 3.29. From this Proposition, we can easily prove Proposition 3.30 and Corollary 3.31. The local product structure is deduced from these in a straightforward fashion - see, e.g., §3 in Cli23].

We begin by showing how to obtain a compact subset of $V_{x}^{u}$ which satisfies the necessary properties to apply the results of 3.3 .1 while still allowing us to carry out the necessary estimates later.
Proposition 3.27. Let $c \in\left(0, \frac{\sup _{\gamma \in G S} \lambda(\gamma)}{4}\right)$ and recall that we've chosen $z_{0} \in G S$ such that $\lambda\left(z_{0}\right)>3 c$. Let $x, y \in \mathcal{B}\left(n_{0}, \varepsilon\right)$ be generic points for $\mu$. Then for all $n \geq n_{0}$, and for all $\bar{\delta}>0$, we can find a compact subset $K \subset \operatorname{int} \mathcal{B}\left(n_{0}, \varepsilon\right)$ such that:
(a) Let $\kappa=\kappa\left(z_{0}\right)$ be as in Section 3.3.2 (see the first paragraph). Then, for all $z \in K$, there exists a sequence $t_{k} \rightarrow \infty$ such that $\lambda\left(g_{t_{k}+s} z\right) \geq 2 c$ for all $|s| \leq \kappa$;
(b) $\mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle K, B_{n}^{s}(x, \varepsilon)\right\rangle\right) \geq(1-\bar{\delta}) \mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle\right)$;
(c) $\mu\left(g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle\pi_{x, y} K, B_{n}^{s}(y, \varepsilon)\right\rangle\right) \geq(1-\bar{\delta}) \mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{y}^{u}, B_{n}^{s}(y, \varepsilon)\right\rangle\right)$.

Proof. Recall that we've chosen $\kappa$ small enough so that $\left.\lambda\right|_{B\left(z_{0}, 3 \kappa\right)} \geq \frac{2}{3} \lambda\left(z_{0}\right)>2 c$. Then by the lower Gibbs property, $B\left(z_{0}, \kappa\right)$ has positive measure. Consequently, the set of generic points $G_{\mu} \subset G S$ has the property that all $z \in G_{\mu}$ have a sequence $t_{z, k} \rightarrow \infty$ with $g_{t_{z, k}} z \in B\left(z_{0}, \kappa\right)$.

Let $n \geq n_{0}$ and $\bar{\delta}>0$ be arbitrary. By inner regularity of $\mu$, let $\mathcal{K} \subset \operatorname{int}\left(\mathcal{B}\left(n_{0}, \varepsilon\right)\right) \cap G_{\mu} \cap g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle$ be a compact subset of relative measure at least $1-\bar{\delta}$; that is,

$$
\mu(\mathcal{K}) \geq(1-\bar{\delta}) \mu\left(G_{\mu} \cap g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle\right)=(1-\bar{\delta}) \mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle\right)
$$

Then, define $K_{x}:=[\mathcal{K}, x]$ to be the projection of $\mathcal{K}$ to $V_{x}^{u}$ under the bracket operation. As $[\cdot, x]: \mathcal{B}\left(n_{0}, \varepsilon\right) \rightarrow$ $V_{x}^{u}$ is continuous by Corollary $3.23, K_{x}$ is compact. Now, $g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle K_{x}, B_{n}^{s}(x, \varepsilon)\right\rangle \supset \mathcal{K}$, and so has relative measure at least $1-\bar{\delta}$. Furthermore, it preserves the necessary recurrence properties. This is because any element $w \in g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle K_{x}, B_{n}^{s}(x, \varepsilon)\right\rangle$ is in $W^{c s}(z)$ for some $z \in G_{\mu}$. Therefore, there exists some $r \in \mathbb{R}$ such that for all large enough $t$ and for all $|s| \leq \kappa, d\left(g_{r+t+s} w, g_{t} z\right) \leq 2 \kappa$, and consequently, $\lambda\left(g_{r+t_{z, k}+s} w\right) \geq 2 c$ for the tail of the sequence $t_{z, k} \rightarrow \infty$.

Finally, observe that we could have performed the same procedure with $y$ instead of with $x$, to obtain a compact set $K_{y} \subset V_{y}^{u}$. To obtain the compact set we desire, take $K:=\left(K_{x} \cup \pi_{x, y}^{-1} K_{y}\right)$. Properties (b) and (c) are immediate from the construction of $K$ and the argument on measures above, and the fact that the boundary of $\mathcal{B}\left(n_{0}, \varepsilon\right)$ has zero measure. Property (a) is likewise immediate from the construction for $K_{x}$, as it is for $K_{y}$. Recall that $\pi_{x, y}^{-1}$ is projection along a local center stable leaf. Points in $K_{y}$ have values of $\lambda$ at least $2 c$ at a sequence of times going to infinity. Hence, they never bound a flat strip, and elements in their local center stable leaf eventually go through the same sequence of cone points with the same turning angles, shifted only by some small change in the time parameter. Hence, property (a) holds for $\pi_{x, y}^{-1} K_{y}$ as well.

The local unstable leaves $V_{x}^{u}$ form a measurable partition of $\mathcal{B}\left(n_{0}, \varepsilon\right)$ as discussed in Cli23] or CPZ20]. (The proof of Proposition 3.28 below justifies the measurability.) Hence (see, e.g., [Cli23, Lemma 3.1]), there is a unique system of conditional measures $\mu_{x}^{u}$ on the leaves $V_{x}^{u}$. The space of leaves $\left\{V_{x}^{u}\right\}_{x \in \mathcal{B}\left(n_{0}, \varepsilon\right)}$ can be identified, using the (topological) local product structure, with $V_{x}^{c s}$ for any $x \in \mathcal{B}\left(n_{0}, \varepsilon\right)$. For such an $x$ we define $\mu_{x}^{c s}$ to be the factor measure: $\mu_{x}^{c s}(E):=\mu\left(\bigcup_{y \in E} V_{y}^{u}\right)$.
Remark. $\left(V_{x}^{c s}, \mu_{x}^{c s}\right)$ is (after renormalization) Lebesgue space, as $V_{x}^{c s}$ is a complete, separable metric space. We note that from the definition of $\mu_{x}^{c s}$ and the flow-invariance of $\mu, \mu_{x}^{c s}$ has no atoms. Indeed, if $\mu_{x}^{c s}\left(\left\{y^{*}\right\}\right)>$ 0 for some $y^{*} \in V_{x}^{c s}$, then for all small $t, \mu\left(g_{t} V_{y^{*}}^{u}\right)=\mu\left(V_{y^{*}}^{u}\right)=\mu_{x}^{c s}\left(\left\{y^{*}\right\}\right)>0$. But for sufficiently small $t$, $s$, $g_{t} V_{y^{*}}^{u}$ and $g_{s} V_{y^{*}}^{u}$ are disjoint. Therefore, for small $\delta>0, \mu\left(\bigcup_{t \in(-\delta, \delta)} g_{t} V_{y^{*}}^{u}\right)=\infty$, a contradiction.
Proposition 3.28. There exists $R^{\prime} \subset \mathcal{B}\left(n_{0}, \varepsilon\right)$ with $\mu\left(B\left(n_{0}, \varepsilon\right) \backslash R^{\prime}\right)=0$ such that for all $x \in R^{\prime}$ and every continuous $\psi: \mathcal{B}\left(n_{0}, \varepsilon\right) \rightarrow \mathbb{R}$,

$$
\int_{V_{x}^{u}} \psi d \mu_{x}^{u}=\lim _{n \rightarrow \infty} \frac{1}{\mu\left(\mathcal{B}\left(n_{0}, \varepsilon\right) \cap g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle\right)} \int_{\left.\mathcal{B}\left(n_{0}, \varepsilon\right) \cap g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle\right)} \psi d \mu
$$

Proof. By CPZ20, Proposition 8.2] it suffices to observe that $\mathcal{B}\left(n_{0}, \varepsilon\right) \cap g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle$ is a sequence of elements of a refining sequence of measurable partitions. For this, define the partition $\xi_{n}=\xi_{n-1} \vee\left\{\mathcal{B}\left(n_{0}, \varepsilon\right) \cap\right.$ $\left.g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle, \mathcal{B}\left(n_{0}, \varepsilon\right) \backslash g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle\right\}$.

Proposition 3.29. There exists $K>0$ such that for almost every $x, y \in \mathcal{B}\left(n_{0}, \varepsilon\right)$ and for every uniformly continuous $\psi: \mathcal{B}\left(n_{0}, \varepsilon\right) \rightarrow(0, \infty)$,

$$
\int_{V_{y}^{u}} \psi d\left(\pi_{x, y}\right)_{*} \mu_{x}^{u} \leq K \int_{V_{y}^{u}} \psi d \mu_{y}^{u}
$$

Proof. Let $\psi: \mathcal{B}\left(n_{0}, \varepsilon\right) \rightarrow(0, \infty)$ be uniformly continuous, and let $\alpha>0$ be small enough so that if $d_{G S}\left(u, u^{\prime}\right) \leq 3 \alpha$, then $\frac{1}{2} \psi\left(u^{\prime}\right) \leq \psi(u) \leq 2 \psi\left(u^{\prime}\right)$. Now let $R^{\prime} \subset \mathcal{B}\left(n_{0}, \varepsilon\right)$ be as in Proposition 3.28, and let $x, y \in R^{\prime} \cap g_{\left(-\frac{1}{n_{0}}, \frac{1}{n_{0}}\right)} R(\varepsilon)$ be generic points for $\mu$. Observe that this is a full measure set, from applying Proposition 3.28 and the fact that $\mu\left(g_{ \pm \frac{1}{n_{0}}} R(\varepsilon)\right)=0$, because $\mu$ is measure-preserving and $\frac{1}{n_{0}}$ is less than one quarter the injectivity radius of $S$. Now, both $\psi \circ \pi_{x, y}$ and $\psi$ are continuous functions, and so we can apply Proposition 3.28 to estimate both

$$
\int_{V_{x}^{u}} \psi \circ \pi_{x, y} d \mu_{x}^{u}=\int_{V_{x}^{u}} \psi d\left(\pi_{x, y}\right)_{*} \mu_{y}^{u} \quad \text { and } \quad \int_{V_{y}^{u}} \psi d \mu_{y}^{u}
$$

To do so, we will always assume that we are working with large enough $n$ so that $g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle \subset$ $\mathcal{B}\left(n_{0}, \varepsilon\right)$ and similarly for $y$. We will now proceed with our proof by using Propositions 3.27 and 3.12 to partition most of $V_{x}^{u}$ and $V_{y}^{u}$ into well-understood sets, for which we have nice upper and lower bounds from the Gibbs property. By Lemma 3.22 , there exists $\rho \in\left[-\frac{2}{n_{0}}, \frac{2}{n_{0}}\right]$ such that $\pi_{x, y}(z):=\left\langle g_{\rho} z, y\right\rangle$ for all $z \in V_{x}^{u}$.

Fix $\rho \in\left[\frac{-2}{n_{0}}, \frac{2}{n_{0}}\right]$ as in Lemma 3.22 . Let $0<\beta<\min \left\{1,\|\psi\|^{-1}\right\}$, and let $c \in\left(0, \frac{\|\lambda\|_{\infty}}{4}\right)$ be chosen so that $\mu\left(\lambda^{-1}([c, \infty))>\frac{3}{4}\right.$. Such a $c$ exists because $\mu\left(\lambda^{-1}(0)\right)=0$ by Corollary 3.8 . For all $n \in \mathbb{N}$, let $K_{n} \subset V_{x}^{u}$ be as in Proposition 3.27 for $c$ with

$$
\mu\left(g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle K_{n}, B_{n}^{s}(x, \varepsilon)\right\rangle\right) \geq\left(1-\beta^{2}\right) \mu\left(g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle\right)
$$

and similarly for $\pi_{x, y} K_{n}$. Then, use Proposition 3.12 to define a partition $\omega_{n}$ of $K_{n}$, such that:

- For each $A_{i} \in \omega_{n}$ there exists $z_{i} \in A_{i}$ and $t_{i}>t_{R}+\frac{2}{n_{0}}$ such that $K_{n} \cap B_{t_{i}}^{u}\left(z_{i}, \frac{\varepsilon}{2}\right) \subset A_{i} \subset B_{t_{i}}^{u}\left(z_{i}, 2 \varepsilon\right)$
- $\lambda\left(g_{t_{i}+r} z_{i}\right) \geq c$ for all $r \in\left[-\frac{4}{n_{0}}, \frac{4}{n_{0}}\right]$
- $\max \left\{\operatorname{diam} A \mid A \in \omega_{n}\right\} \leq \min \left\{\alpha e^{-2|\rho|}, d\left(K_{n}, \partial \mathcal{B}\left(n_{0}, \varepsilon\right)\right)\right\}$.

The second condition follows from choosing $n_{0}$ large enough so that $\frac{4}{n_{0}} \leq \kappa$ in the statement of Proposition 3.12 which in turn relies only on the choice of $\kappa$ in Proposition 3.27. The condition on the diameter of $\omega_{n}$ follows from Corollary 3.15, as $K_{n} \subset \operatorname{Reg}$, while $\mathrm{NE}(\varepsilon) \subset \operatorname{Sing}$ due to [CCE ${ }^{+} 23$, Lemma 2.16]. This partition will allow us to estimate $\int_{V_{x}^{u}} \psi \circ \pi_{x, y} d \mu_{x}^{u}$, but it will not help with $\int_{V_{y}^{u}} \psi d \mu_{y}^{u}$, and so we define an auxiliary partition of $\pi_{x, y} K_{n}$ by $\omega_{n}^{\prime}:=\left[\omega_{n}, y\right]=\left\langle g_{\rho} \omega_{n}, y\right\rangle$. This auxiliary partition has the following properties:

- For each $A_{i}^{\prime} \in \omega_{n}^{\prime}$, taking $z_{i}^{\prime}:=\left\langle g_{\rho} z_{i}, y\right\rangle$, we have

$$
\pi_{x, y} K_{n} \cap B_{t_{i}+|\rho|}^{u}\left(z_{i}^{\prime}, \frac{\varepsilon}{2}\right) \subset A_{i}^{\prime} \subset B_{t_{i}-|\rho|}^{u}\left(z_{i}^{\prime}, 2 \varepsilon\right)
$$

- $\lambda\left(g_{t_{i}+|\rho|} z_{i}^{\prime}\right) \geq c$
- $\max \left\{\operatorname{diam} A^{\prime} \mid A^{\prime} \in \omega_{n}^{\prime}\right\} \leq \alpha$.

The first property of $\omega_{n}^{\prime}$ follows from Lemmas 3.24 and 3.25 The proof of the second inclusion is as follows, and the first is similar:

$$
A_{i}^{\prime}=\left\langle g_{\rho} A_{i}, y\right\rangle \subset\left\langle g_{\rho} B_{t_{i}}^{u}\left(z_{i}, \varepsilon\right), y\right\rangle \subset\left\langle B_{t_{i}-|\rho|}^{u}\left(g_{\rho} z_{i}, \varepsilon\right), y\right\rangle=B_{t_{i}-|\rho|}^{u}\left(z_{i}^{\prime}, \varepsilon\right)
$$

For the second property, applying Lemma 2.32, we have

$$
\lambda\left(g_{t_{i}+|\rho|} z_{i}^{\prime}\right)=\lambda\left(g_{t_{i}+|\rho|}\left\langle g_{\rho} z_{i}, y\right\rangle\right)=\lambda\left(g_{t_{i}+|\rho|+\rho} z_{i}\right)
$$

Recall that while Lemma 2.32 requires $t_{i}$ to be large enough, dependent on the value of $c$, Proposition 3.27 allows us to choose $t_{i}$ as large as we would like. Now, simply note that $\left|t_{i}+|\rho|+\rho-t_{i}\right| \leq \frac{4}{n_{0}}$. Finally, the third property follows from Lemma 3.22 Using these partitions, we can carry out our estimates as desired.

Take $n$ large enough so that $\frac{2}{n} \leq \alpha$ and so that both $\operatorname{diam} B_{n}^{s}(x, \varepsilon) \leq \alpha$ and $\operatorname{diam} B_{n}^{s}(y, \varepsilon) \leq \alpha$. Then for $A \in \omega_{n}$,

$$
\operatorname{diam} g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle A, B_{n}^{s}(x, \varepsilon)\right\rangle \leq \frac{2}{n}+\operatorname{diam} A+\operatorname{diam} B_{n}^{s}(x, \varepsilon) \leq 3 \alpha
$$

Similarly for $A^{\prime} \in \omega_{n}^{\prime}$, $\operatorname{diam} g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle A^{\prime}, B_{n}^{s}(y, \varepsilon)\right\rangle \leq 3 \alpha$.

Now, we will find an upper bound on $\int_{V_{x}^{u}} \psi \circ \pi_{x, y} d \mu_{x}^{u}$. For all large enough $n$ (as described above), we have

$$
\begin{aligned}
\int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle
\end{aligned} \psi \circ \pi_{x, y} d \mu-\beta^{2}\|\psi\| \mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle\right) \leq \int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle K_{n}, B_{n}^{s}(x, \varepsilon)\right\rangle} \psi \circ \pi_{x, y} d \mu
$$

The final step uses Lemma 3.20. We will also need the following lower bound:

$$
\begin{aligned}
\mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle\right) & =\mu\left(g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u} \backslash K_{n}, B_{n}^{s}(x, \varepsilon)\right\rangle\right)+\sum_{i=1}^{k} \mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle A_{i}, B_{n}^{s}(x, \varepsilon)\right\rangle\right) \\
& \geq \mu\left(g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u} \backslash K_{n}, B_{n}^{s}(x, \varepsilon)\right\rangle\right)+\sum_{i=1}^{k} \mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle K_{n} \cap B_{t_{i}}^{u}\left(z_{i}, \frac{\varepsilon}{2}\right), B_{n}^{s}\left(x, \frac{\varepsilon}{2}\right)\right\rangle\right) \\
& \geq \sum_{i=1}^{k} \mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle B_{t_{i}}^{u}\left(z_{i}, \frac{\varepsilon}{2}\right), B_{n}^{s}\left(x, \frac{\varepsilon}{2}\right)\right\rangle\right) \\
& \geq \sum_{i=1}^{k} \mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle B_{t_{i}}^{u}\left(z_{i}, \frac{\varepsilon}{2}\right), B_{n}^{s}\left(x, \frac{\varepsilon}{2}\right)\right\rangle\right)
\end{aligned}
$$

From this, using Lemma 3.21, it follows that

$$
\mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle\right) \geq \frac{n_{0}}{n\left(n_{0} \varepsilon+1\right)} \sum_{i=1}^{k} Q_{2}\left(\frac{\varepsilon}{2}, \lambda\left(g_{t_{i}} z_{i}\right), \lambda\left(g_{-n} x\right)\right) e^{-\left(n+t_{i}\right) P(\phi)+\int_{-n}^{t_{i}} \phi\left(g_{r} z_{i}\right) d r}
$$

Finally, we will need the following computations using the Bowen property to make use of these results. Recall that $\phi$ has the Bowen property at scale $\varepsilon_{0}$, and that $\operatorname{diam} \mathcal{B}\left(n_{0}, \varepsilon\right)<\varepsilon_{0}$. Now, observe that $d_{G S}\left(g_{r} z, g_{r} x\right)$ decreases monotonically to 0 as $r \rightarrow-\infty$ for all $z \in V_{x}^{u}$, while $d_{G S}\left(g_{r} z, g_{r} \pi_{x, y} z\right)$ decreases monotonically to $|\rho|$. Therefore, for any such $z$, we have $z \in B_{[-n, 0]}\left(x, \varepsilon_{0}\right)$ and $z \in B_{[0, n]}\left(\pi_{x, y} z, \varepsilon_{0}-\rho\right)$ for all $n \geq 0$. Applying the Bowen property via Proposition 3.6, there exists $L>0$ such that for all $n_{1}, n_{2} \geq 0$,

$$
\left|\int_{-n_{1}}^{n_{2}} \phi\left(g_{r} z\right) d r\right| \leq\left|\int_{-n_{1}}^{0} \phi\left(g_{r} z\right) d r-\int_{-n_{1}}^{0} \phi\left(g_{r} x\right) d r\right|+\left|\int_{0}^{n_{2}} \phi\left(g_{r} z\right) d r-\int_{0}^{n_{2}} \phi\left(g_{r} \pi_{x, y} z\right) d r\right| \leq 2 L
$$

Hence, for any $n$ for which $\lambda\left(g_{-n} x\right) \geq c$, we have that (taking $z_{i}$ as our elements of $V_{x}^{u}$ to apply the Bowen property), we have, writing $Q_{2}:=Q_{2}\left(\frac{\varepsilon}{2}, c, c\right)$

$$
\begin{aligned}
\frac{\int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle} \psi \circ \pi_{x, y} d \mu}{\mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle\right)} & \leq \beta+\frac{\sum_{i=1}^{k} 2 \psi\left(z_{i}^{\prime}\right) \frac{n_{0}}{n} Q_{1} e^{-\left(n+t_{i}\right) P(\phi)+\int_{-n}^{t_{i}} \phi\left(g_{r}\left\langle z_{i}, x\right)\right) d r}}{\frac{n_{0}}{n\left(n_{0} \varepsilon+1\right)} \sum_{i=1}^{k} Q_{2}\left(\frac{\varepsilon}{2}, \lambda\left(g_{t_{i}} z_{i}\right), \lambda\left(g_{-n} x\right)\right) e^{-\left(n+t_{i}\right) P(\phi)+\int_{-n}^{t_{i}} \phi\left(g_{r}\left\langle z_{i}, x\right\rangle\right) d r}} \\
& \leq \beta+\frac{\left(n_{0} \varepsilon+1\right) e^{2 L+\int_{-n}^{0} \phi\left(g_{r} x\right) d r} \sum_{i=1}^{k} 2 \psi\left(z_{i}^{\prime}\right) Q_{1} e^{-t_{i} P(\phi)+\int_{0}^{t_{i}} \phi\left(g_{r} \pi_{x, y} z_{i}\right) d r}}{e^{-2 L+\int_{-n}^{o} \phi\left(g_{r} x\right) d r} \sum_{i=1}^{k} Q_{2} e^{-t_{i} P(\phi)+\int_{0}^{t_{i}} \phi\left(g_{r} \pi_{x, y} z_{i}\right) d r}} \\
& =\beta+\frac{e^{4 L} Q_{1}}{Q_{2}} \frac{\left(n_{0} \varepsilon+1\right) \sum_{i=1}^{k} 2 \psi\left(z_{i}^{\prime}\right) e^{-t_{i} P(\phi)+\int_{0}^{t_{i}} \phi\left(g_{r} \pi_{x, y} z_{i}\right) d r}}{\sum_{i=1}^{k} e^{-t_{i} P(\phi)+\int_{0}^{t_{i}} \phi\left(g_{r} \pi_{x, y} z_{i}\right) d r}} .
\end{aligned}
$$

Note that $\beta$ does not depend on our choice of $n$, although the rest of the computation does, as our collection of $z_{i}$ and $t_{i}$ depends on our choice of $K_{n}$.

Now, we will carry out similar computations to find a lower bound for the corresponding estimates of $\int_{V_{u}^{u}} \psi d \mu_{y}^{u}$. These mirror those carried out above, except using the lower Gibbs bound instead of the upper Gibbs bound and vice versa. The main difference is that instead of partitioning $K_{n} \subset V_{x}^{u}$ for each $n$, we will instead need to use the partition $\omega_{n}^{\prime}:=\pi_{x, y} \omega_{n}=\left\langle g_{\rho} \omega_{n}, y\right\rangle$ whose properties we have outlined above, which will be a partition of $\pi_{x, y} K_{n} \subset V_{y}^{u}$. We restate them here:

- For each $A_{i}^{\prime} \in \omega_{n}^{\prime}$, taking $z_{i}^{\prime}:=\pi_{x, y} z_{i}=\left\langle g_{\rho} z_{i}, y\right\rangle$, we have $\pi_{x, y} K_{n} \cap B_{t_{i}+|\rho|}^{u}\left(z_{i}^{\prime}, \frac{\varepsilon}{2}\right) \subset A_{i}^{\prime} \subset$ $B_{t_{i}-|\rho|}^{u}\left(z_{i}^{\prime}, 2 \varepsilon\right)$
- $\lambda\left(g_{t_{i}+|\rho|} z_{i}^{\prime}\right) \geq c$
- $\max \left\{\operatorname{diam} A^{\prime} \mid A^{\prime} \in \omega_{n}^{\prime}\right\} \leq \alpha$.

Recalling that we work with $n$ large enough so that $\operatorname{diam} g_{\left[-\frac{1}{n}, \frac{1}{n}\right]}\left\langle A_{i}^{\prime}, B_{n}^{s}(y, \varepsilon)\right\rangle \leq 3 \alpha$, and assuming that $n$ is chosen so that $\lambda\left(g_{-n} y\right) \geq c$, we obtain a lower bound as follows. First, observe
$\left.\left.\sum_{i=1}^{k} \int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle B_{t_{i}+|\rho|}^{u}\left(z_{i}^{\prime}, \frac{\varepsilon}{2}\right) \cap \pi_{x, y} K_{n}, B_{n}^{s}(y, \varepsilon)\right\rangle} \psi d \mu \geq \sum_{i=1}^{k} \int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle B_{t_{i}+|\rho|}^{u}\right.}\left\langle z_{i}^{\prime}, \frac{\varepsilon}{2}\right), B_{n}^{s}(y, \varepsilon)\right\rangle\right)<\int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{y}^{u} \backslash \pi_{x, y} K_{n}, B_{n}^{s}(y, \varepsilon)\right\rangle} \psi d \mu$.
Consequently, we have

$$
\begin{aligned}
2 \int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{y}^{u}, B_{n}^{s}(y, \varepsilon)\right\rangle} \psi d \mu & \geq \int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle\pi_{x, y} K_{n}, B_{n}^{s}(y, \varepsilon)\right\rangle} \psi d \mu+\int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}}\left\langle V_{y}^{u} \backslash \pi_{x, y} K_{n}, B_{n}^{s}(y, \varepsilon)\right\rangle \\
& =\sum_{i=1}^{k} \int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle A_{i}^{\prime}, B_{n}^{s}(y, \varepsilon)\right\rangle} \psi d \mu+\int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{y}^{u} \backslash \pi_{x, y} K_{n}, B_{n}^{s}(y, \varepsilon)\right\rangle} \psi d \mu \\
& \geq \sum_{i=1}^{k} \int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle B_{t_{i}+|\rho|}^{u}\left(z_{i}^{\prime}, \frac{\varepsilon}{2}\right), B_{n}^{s}(y, \varepsilon)\right\rangle} \psi d \mu \\
& \geq \sum_{i=1}^{k} \int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle B_{t_{i}+|\rho|}^{u}\left(z_{i}^{\prime}, \frac{\varepsilon}{2}\right), B_{n}^{s}(y, \varepsilon)\right\rangle} \frac{\psi\left(z_{i}^{\prime}\right)}{2} d \mu \\
& =\sum_{i=1}^{k} \frac{\psi\left(z_{i}^{\prime}\right)}{2} \mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle B_{t_{i}+|\rho|}^{u}\left(z_{i}^{\prime}, \frac{\varepsilon}{2}\right), B_{n}^{s}(y, \varepsilon)\right\rangle\right) \\
& \geq \sum_{i=1}^{k} \frac{\psi\left(z_{i}^{\prime}\right)}{2} \frac{n_{0}}{n\left(n_{0} \varepsilon+1\right)} Q_{2}\left(\frac{\varepsilon}{2}, \lambda\left(\left.g_{t_{i}+|\rho|}\right|_{i} ^{\prime}\right), \lambda\left(g_{n} y\right)\right) e^{\left.-\left(n+t_{i}+|\rho|\right) P(\phi)+\int_{-n}^{t_{i}+|\rho|} \phi\left(g_{r}\left\langle z_{i}^{\prime}, y\right\rangle\right)\right) d r} \\
& \geq Q_{2} \frac{n_{0}}{n\left(n_{0} \varepsilon+1\right)} e^{-L} e^{\int_{-n}^{0} \phi\left(g_{r} y\right) d r} \sum_{i=1}^{k} \frac{\psi\left(z_{i}^{\prime}\right)}{2} e^{-\left(n+t_{i}+|\rho|\right) P(\phi)+\int_{0}^{t_{i}+|\rho|} \phi\left(g_{r}\left\langle\left\langle z_{i}^{\prime}, y\right\rangle\right) d r\right.} \\
& \geq Q_{2} \frac{n_{0}}{n\left(n_{0} \varepsilon+1\right)} e^{-L} e^{\int_{-n}^{0} \phi\left(g_{r} y\right) d r} e^{-|\rho|(P(\phi)+\|\phi\|)} \sum_{i=1}^{k} \frac{\psi\left(z_{i}^{\prime}\right)}{2} e^{-\left(n+t_{i}\right) P(\phi)+\int_{0}^{t_{i}} t_{i} \phi\left(g_{r} z_{i}^{\prime}\right) d r}
\end{aligned}
$$

where the second to last inequality utilizes the Bowen property as we did in our previous estimates.
We also have the following upper bound:

$$
\begin{aligned}
\mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{y}^{u}, B_{n}^{s}(y, \varepsilon)\right\rangle\right) & \leq \frac{1}{1-\beta^{2}} \sum_{i=1}^{k} \mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle A_{i}^{\prime}, B_{n}^{s}(y, \varepsilon)\right\rangle\right) \\
& \leq \frac{1}{1-\beta^{2}} \sum_{i=1}^{k} \mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle B_{t_{i}-|\rho|}^{u}\left(z_{i}^{\prime}, 2 \varepsilon\right), B_{n}^{s}(y, 2 \varepsilon)\right\rangle\right) \\
& \leq \frac{1}{1-\beta^{2}} \frac{n_{0}}{n} \sum_{i=1}^{k} Q_{1} e^{-\left(n+t_{i}-|\rho|\right) P(\phi)+\int_{-n}^{t_{i}-|\rho|} \phi\left(g_{r}\left\langle z_{i}^{\prime}, y\right\rangle\right) d r} \\
& \leq \frac{1}{1-\beta^{2}} \frac{n_{0}}{n} e^{L} e^{\int_{-n}^{0} \phi\left(g_{r} y\right) d r} e^{|\rho|(P(\phi)+\|\phi\|)} \sum_{i=1}^{k} Q_{1} e^{-\left(n+t_{i}\right) P(\phi)+\int_{0}^{t_{i}} \phi\left(g_{r} z_{i}^{\prime}\right) d r}
\end{aligned}
$$

Combining these two, for all large enough $n$ such that $\lambda\left(g_{-n} y\right) \geq c$, we have that

$$
\begin{aligned}
& \frac{\int_{\mathcal{B}\left(n_{0}, \varepsilon\right) \cap g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{y}^{u}, B_{n}^{s}(y, \varepsilon)\right\rangle} \psi d \mu}{\left.\mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]} V_{y}^{u}, B_{n}^{s}(y, \varepsilon)\right\rangle\right)} \\
& \quad \geq\left(1-\beta^{2}\right) \frac{Q_{2} \frac{n_{0}}{2 n\left(n_{0} \varepsilon+1\right)} e^{-L} e^{\int_{-n}^{0} \phi\left(g_{r} y\right) d r} e^{-|\rho|(P(\phi)+\|\phi\|)} \sum_{i=1}^{k} \frac{\psi\left(z_{i}^{\prime}\right)}{2} e^{-\left(n+t_{i}\right) P(\phi)+\int_{0}^{t_{i}} \phi\left(g_{r} z_{i}^{\prime}\right) d r}}{\frac{n_{0}}{n} e^{L} e^{\int_{-n}^{0} \phi\left(g_{r} y\right) d r} e^{|\rho|(P(\phi)+\|\phi\|)} \sum_{i=1}^{k} Q_{1} e^{-\left(n+t_{i}\right) P(\phi)+\int_{0}^{t_{i}} \phi\left(g_{r} z_{i}^{\prime}\right) d r}} \\
& \quad=\left(1-\beta^{2}\right) \frac{Q_{2} e^{-2|\rho|(P(\phi)-\|\phi\|)}}{2 Q_{1} e^{2 L}} \frac{\sum_{i=1}^{k} \frac{\psi\left(z_{i}^{\prime}\right)}{2} e^{-t_{i} P(\phi)+\int_{0}^{t_{i}} \phi\left(g_{r} z_{i}^{\prime}\right) d r}}{\left(n_{0} \varepsilon+1\right) \sum_{i=1}^{k} e^{-t_{i} P(\phi)+\int_{0}^{t_{i}} \phi\left(g_{r} z_{i}^{\prime}\right) d r} .}
\end{aligned}
$$

Therefore, for all large $n$ for which both $\lambda\left(g_{-n} x\right)$ and $\lambda\left(g_{-n} y\right) \geq c$, we can now directly compare our estimates for $\int_{V_{x}^{u}} \psi \circ \pi_{x, y} d \mu_{x}^{u}$ and a lower bound for $\int_{V_{y}^{u}} \psi d \mu_{y}^{u}$. As we have chosen $c$ small enough so that $\mu\left(\lambda^{-1}([c, \infty))\right)>\frac{3}{4}$, there exists a sequence of such $n \rightarrow \infty$, because $x$ and $y$ are both generic points for $\mu$. In particular, for such a choice of $n$, recall that we have already shown that

$$
\frac{\sum_{i=1}^{k} 2 \psi\left(z_{i}^{\prime}\right) e^{-t_{i} P(\phi)+\int_{0}^{t_{i}} \phi\left(g_{r} \pi_{x, y} z_{i}\right) d r}}{\sum_{i=1}^{k} e^{-t_{i} P(\phi)+\int_{0}^{t_{i}} \phi\left(g_{r} \pi_{x, y} z_{i}\right) d r}} \geq \frac{Q_{2}}{Q_{1} e^{4 L}\left(n_{0} \varepsilon+1\right)}\left(\frac{\int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle} \psi \circ \pi_{x, y} d \mu}{\mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle\right)}-\beta\right)
$$

Consequently, we have:

$$
\begin{aligned}
\frac{\int_{\mathcal{B}\left(n_{0}, \varepsilon\right) \cap g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}}\left\langle V_{y}^{u}, B_{n}^{s}(y, \varepsilon)\right\rangle}{} \psi d \mu & \geq\left(1-\beta^{2}\right) \frac{Q_{2} e^{-2|\rho|(P(\phi)+\|\phi\|}}{2 Q_{1} e^{2 L}\left(n_{0} \varepsilon+1\right)} \frac{\left.\sum_{i=1}^{k} \frac{\psi\left(z_{i}^{\prime}\right)}{2} e^{-t_{i} P(\phi)+\int_{0}^{t_{i}}}\left\langle V_{y}^{u}, B_{n}^{s}(y, \varepsilon)\right\rangle\right)}{\sum_{i=1}^{k} e^{\left.-t_{i} P(\phi)+\int_{r} z_{i}^{\prime}\right) d r}}{ }^{t_{i}} \phi\left(g_{r} z_{i}^{\prime}\right) d r \\
& \geq\left(1-\beta^{2}\right) \frac{Q_{2}^{2} e^{-2|\rho|(P(\phi)+\|\phi\|)}}{2 Q_{1}^{2} e^{6 L}\left(n_{0} \varepsilon+1\right)^{2}}\left(\frac{\int_{g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle} \psi \circ \pi_{x, y} d \mu}{\mu\left(g_{\left[\frac{-1}{n}, \frac{1}{n}\right]}\left\langle V_{x}^{u}, B_{n}^{s}(x, \varepsilon)\right\rangle\right)}-\beta\right) .
\end{aligned}
$$

As $\beta$ does not depend on $n$ or on $c$, we can make it arbitrarily small. Thus, we have shown that for an appropriate subsequence $n_{k} \rightarrow \infty$, we have

$$
\begin{aligned}
& \leq e^{6 L}\left(\frac{2 Q_{1}}{Q_{2}\left(n_{0} \varepsilon+1\right)}\right)^{2} e^{2|\rho|(P(\phi)+\|\phi\|)} .
\end{aligned}
$$

Using Proposition 3.29, we now complete our proof of local product structure. This proof is standard, and follows Cli23, Theorem 1.2].
Proposition 3.30. Given $\mathcal{B}\left(n_{0}, \varepsilon\right) \subset \operatorname{Reg}(c)$ for some $c>0$, there exists $K:=K(c)>0$ such that for almost every $x, y \in \mathcal{B}\left(n_{0}, \varepsilon\right)$,

$$
K^{-1} \leq \frac{d\left(\pi_{x, y}\right)_{*} \mu_{x}^{u}}{d \mu_{y}^{u}} \leq K .
$$

Proof. Let $U \subset V_{y}^{u}$ be an arbitrary open set, and observe that there exists a sequence of continuous functions $\psi_{n}: V_{x}^{u} \rightarrow(0,1]$ which converges pointwise to $\mathbf{1}_{U}$. For each such $\psi_{n}$ we have by Proposition 3.29 that

$$
\int_{V_{y}^{u}} \psi_{n} d\left(\pi_{x, y}\right)_{*} \mu_{x}^{u} \leq K \int_{V_{y}^{u}} \psi_{n} d \mu_{y}^{u} .
$$

Then, by dominated convergence, this implies that $\left(\pi_{x, y}\right)_{*} \mu_{x}^{u}(U) \leq K \mu_{y}^{u}(U)$. Letting $E \subset V_{y}^{u}$ be an arbitrary measurable set, by outer regularity, we have that

$$
\mu_{x}^{u}\left(\pi_{x, y}^{-1} E\right)=\inf \left\{\mu_{x}^{u}\left(\pi_{x, y}^{-1} U \mid U \supset E \text { is open }\right\} \leq \inf \left\{K \mu_{y}^{u}(U \mid U \supset E \text { is open }\}=K \mu_{y}^{u}(E) .\right.\right.
$$

As $x$ and $y$ are interchangeable, we also have the reverse inequality, and so our proof is complete.
Corollary 3.31. For almost every $\gamma \in G S$, there exists a flow box $\mathcal{B}\left(n_{0}, \varepsilon\right) \ni \gamma$ and some $K:=K(\gamma)>0$ such that for $\mu$-almost every $x, y \in \mathcal{B}\left(n_{0}, \varepsilon\right)$,

$$
K^{-1} \leq \frac{d\left(\pi_{x, y}\right)_{*} \mu_{x}^{u}}{d \mu_{y}^{u}} \leq K .
$$

Proof. Recall that if $\lambda(\gamma)=0$, then $\lambda\left(g_{t} \gamma\right)=0$ for all $t \geq 0$ or all $t \leq 0$ by [CE ${ }^{+23}$, Proposition 3.4]. As $\mu\left(\lambda^{-1}(0)\right)=0$ by Corollary 3.8, this implies that for $\mu$-a.e. $\gamma \in G S, \lambda\left(g_{t} \gamma\right)>0$ for all $t \in \mathbb{R}$. Consequently, we can construct a rectangle at $\mu$-a.e. $\gamma$ via Lemma 2.27 and in turn, a flow box $\mathcal{B}\left(n_{0}, \varepsilon\right)$ for some appropriate choice of $n_{0}$ and $\varepsilon$. Applying Proposition 3.30 completes the proof.

## 4. The Bernoulli property

The proof of the Bernoulli property (TheoremB) follows an argument due to Ornstein and Weiss OW73. A presentation of their argument with somewhat more detail can be found in CH96; we follow this presentation most closely.

After some preliminary definitions and lemmas, we provide an outline of the argument before beginning the proof.
4.1. Definitions and lemmas. We begin by recording a few definitions and lemmas needed in this section alone.

Definition 4.1. Given a measurable set $A \subset G S$ and $x \in A, V_{x}^{u, A}$ is the connected component of $W^{u}(x) \cap A$ containing $x$.

Lemma 4.2. Let $c>0$ and $B>0$ be given. Then there exists some $\delta$, depending only on $c$ and $B$, such that the following holds:

Let $q \in G S$ be such that $\lambda\left(g_{\hat{t}} q\right)>c$ for some $\hat{t} \in\left[\frac{\theta_{0}}{2 c}+2, B\right]$. If then $x, y \in B(q, \delta)$ and $y \in V_{x}^{u, B(q, \delta)}$, then $x\left(-\infty, t_{0}\right]=y\left(-\infty, t_{0}\right]$ for some $t_{0} \geq 0$.

Proof. Suppose $\lambda\left(g_{\hat{t}} q\right)>c>0$ for some $\hat{t} \in\left[\frac{\theta_{0}}{2 c}+2, B\right]$. By $\mathrm{CCE}^{+} 23$, Prop 3.9], there exists $t_{1} \in\left[\hat{t}-\frac{\theta_{0}}{2 c}, \hat{t}+\right.$ $\left.\frac{\theta_{0}}{2 c}\right] \subset\left[2, B+\frac{\theta_{0}}{2 c}\right]$ such that $q\left(t_{1}\right)$ is a cone point at which $q$ turns by angle at least $s_{0} c$ away from $\pm \pi$.

Using the uniform continuity of the geodesic flow over $t \in\left[0, B+\frac{\theta_{0}}{2 c}+2\right]$, choose $\delta$ such that

- $\delta<\frac{s_{0} c}{16}$, and
- if $d_{G X}\left(z, z^{\prime}\right)<\delta$, then $d_{G X}\left(g_{t} z, g_{t} z^{\prime}\right)<\frac{s_{0} c}{16}$ for all $t \in\left[0, B+\frac{\theta_{0}}{2 c}+2\right]$.

Note that $\delta$ depends only on $c$ and $B$.
By Lemma 2.8, since $d_{G S}(x, q)<\delta<\frac{s_{0} c}{16}$ and $d_{G S}\left(g_{t_{1}+2} x, g_{t_{1}+2} q\right)<\delta<\frac{s_{0} c}{16}, d_{S}(x(0), q(0))<\frac{s c}{8}$ and $d_{S}\left(x\left(t_{1}+2\right), q\left(\overline{t_{1}}+2\right)\right)<\frac{s c}{8}$. By the $\mathrm{CAT}(0)$ condition and some simple Euclidean trigonometry, the angle at $q\left(t_{1}\right)$ between $q\left(0, t_{1}\right)$ and the segment $\left[x(0), q\left(t_{1}\right)\right]$ is $<\frac{s_{0} c}{2}$. Similarly, the angle at $q\left(t_{1}\right)$ between $q\left(t_{1}, t_{1}+1\right)$ and the segment $\left[q\left(t_{1}\right), x\left(t_{1}+2\right)\right.$ is also $<\frac{s_{0} c}{2}$.

The angle that $q$ turns with at $q\left(t_{1}\right)$ is at least $s_{0} c$ away from $\pm \pi$. Therefore, the angle between the segments $\left[x(0), q\left(t_{1}\right)\right]$ and $\left[q\left(t_{1}\right), x\left(t_{1}+2\right)\right]$ is $>\pi$ on both sides, and so the concatenation of these geodesic segments is itself a geodesic segment. Therefore, the cone point $q\left(t_{1}\right)$ belongs to the geodesic $x$. Say it is $x\left(t_{0}\right)$. By choosing $\delta \ll 1$, we can ensure that $t_{0}>0$.

The argument above holds for $y$ as well, so $q\left(t_{1}\right)$ belongs to $y$. Lifting $x$ and $y$ to nearby geodesics in $\tilde{S}$, since $y \in V_{x}^{u, B(q, \delta)}, \tilde{x}(-\infty)=\tilde{y}(-\infty)$. Then $\tilde{x}\left(t_{0}\right)=\tilde{y}\left(t_{0}\right)$, and the Lemma follows immediately.

The following definitions relate to partitions, the main objects of study in this section. For the remainder of this section, let $(X, \mu)$ and $(Y, \nu)$ be general probability measure spaces.

Definition 4.3. A property $\mathcal{P}$ holds for $\varepsilon$-almost every set in a partition $\left\{C_{j}\right\}$ of $(X, \mu)$ if

$$
\mu\left(\bigcup_{j^{\prime} \in J^{\prime}} C_{j^{\prime}}\right)<\varepsilon, \quad \text { where } J^{\prime}=\left\{j \in J: \mathcal{P} \text { does not hold for } C_{j}\right\}
$$

A distance between finite partitions can be defined as follows. (See [PVa19] or CH96].) Let $J(\mu, \nu)$ be the set of joinings of $\mu$ and $\nu$. That is, $J(\mu, \nu)$ is the set of probability measures $\lambda$ on $X \times Y$ such that $\left(\pi_{1}\right)_{*} \lambda=\mu$ and $\left(\pi_{2}\right)_{*} \lambda=\nu$, where $\pi_{i}$ are the usual projections onto the factors or $X \times Y$. Given measurable partitions $\xi=\left\{C_{1}, \ldots, C_{k}\right\}$ of $X$ and $\eta=\left\{D_{1}, \cdots, D_{k}\right\}$ of $Y$, for each $x \in X$, we write $\xi(x)$ for the index $i$ such that $x \in C_{i}$. Similarly $\eta(y)$ is the index $j$ such that $y \in D_{j}$.
Definition 4.4. The $\bar{d}$-distance between $\xi$ and $\eta$ is

$$
\bar{d}(\xi, \eta):=\inf _{\lambda \in J(\mu, \nu)} \lambda(\{(x, y): \xi(x) \neq \eta(y)\})
$$

This distance can be extended to sequences of partitions $\left\{\xi_{i}\right\}_{1}^{n}$ and $\left\{\eta_{i}\right\}_{i}^{n}$ as well.
Definition 4.5. Given two sequences of partitions $\left\{\xi_{i}\right\}_{1}^{n}$ and $\left\{\eta_{i}\right\}_{1}^{n}$, we define

$$
h(x, y):=\frac{1}{n} \sum_{i=1}^{n} \delta\left\{\xi_{i}(x) \neq \eta_{i}(y)\right\}
$$

where $\delta$ is the indicator function.
Definition 4.6. The $\bar{d}$-distance between sequences of partitions $\left\{\xi_{i}\right\}_{1}^{n}$ and $\left\{\eta_{i}\right\}_{1}^{n}$ is

$$
\bar{d}\left(\left\{\xi_{i}\right\}_{1}^{n},\left\{\eta_{i}\right\}_{1}^{n}\right):=\inf _{\lambda \in J(\mu, \nu)} \int h(x, y) d \lambda(x, y)
$$

Definition 4.7. For a measurable $E \subset X$ with $\mu(E)>0$, the conditional measure induced by $\mu$ on $E$ is

$$
\left.\mu\right|_{E}(S):=\frac{\mu(S \cap E)}{\mu(E)}
$$

If $\xi=\left\{C_{1}, \ldots, C_{k}\right\}$ is a partition of $(X, \mu)$, it induces the partition

$$
\left.\xi\right|_{E}:=\left\{C_{1} \cap E, \ldots, C_{k} \cap E\right\}
$$

on $\left(E,\left.\mu\right|_{E}\right)$.

A central tool in the argument below will be Very Weak Bernoulli partitions, introduced by Ornstein in Orn70. Let $f$ be a measure-preserving, invertible map on ( $X, \mu$ ).
Definition 4.8. A finite partition $\xi$ is Very Weak Bernoulli (VWB) if for any $\epsilon>0$, there exists $N \in \mathbb{N}$ such that for all $N_{1} \geq N_{0} \geq N$, for all $n>0$, and for $\epsilon$-almost every $A \in \bigvee_{k=N_{0}}^{N_{1}} f^{k} \xi$, we have

$$
\bar{d}\left(\left\{\left.f^{-i} \xi\right|_{A}\right\}_{1}^{n},\left\{f^{-i} \xi\right\}_{1}^{n}\right) \leq \epsilon
$$

with respect to $\left(A,\left.\mu\right|_{A}\right)$ and $(X, \mu)$.
4.2. An outline of the argument. Our main task in this section is to construct a sequence of VWB partitions with arbitrarily small diameter. To do so, we want to show that we can satisfy the conditions of the following:

Lemma 4.9 (OW73] Lemma 1.3, or CH96] Lemma 4.3). Let $(X, \mu)$ and $(Y, \nu)$ be two non-atomic, Lebesgue probability spaces. Let $\left\{\alpha_{i}\right\},\left\{\beta_{i}\right\}$ for $1 \leq i \leq n$, be two sequences of partitions of $X, Y$, respectively. Suppose there is a map $\psi: X \rightarrow Y$ such that
(1) There is a set $E_{1} \subset X$ with $\mu\left(E_{1}\right)<\varepsilon$ such that for all $x \notin E_{1}$,

$$
h(x, \psi x)<\varepsilon ;
$$

(2) There is a set $E_{2} \subset X$ with $\mu\left(E_{2}\right)<\varepsilon$ such that for any measurable $B \subset X \backslash E_{2}$,

$$
\left|\frac{\mu(B)}{\nu(\psi B)}-1\right|<\varepsilon .
$$

Then $\bar{d}\left(\left\{\alpha_{i}\right\},\left\{\beta_{i}\right\}\right)<c \varepsilon$ (where $c$ is some constant depending only on our underlying system.)
We now revert from the general notation above to our specific situation as outlined in Theorem A replacing the general probability measure space ( $X, \mu$ ) with $(G S, \mu)$, and letting $f=g_{1}$ be the time-one map for the geodesic flow.

Specifically, we will apply Lemma 4.9 with:

- $\left(X, \mu,\left\{\alpha_{i}\right\}\right)=\left(A,\left.\mu\right|_{A},\left\{\left.f^{-i} \xi\right|_{A}\right\}\right)$
- $\left(Y, \nu,\left\{\beta_{i}\right\}\right)=\left(G S, \mu,\left\{f^{-i} \xi\right\}\right)$
for $i=1+m, \ldots, n+m$, for some $m$ to be chosen below. $G S$ is a complete, separable metric space, so these are indeed Lebesgue spaces. Since $\mu$ is flow invariant, $\mu$ and $\left.\mu\right|_{A}$ can have no atoms. Note that to satisfy Definition 4.8 we only need to apply this for $\varepsilon$-almost every $A \in \bigvee_{k=N_{0}}^{N_{1}} f^{k} \xi$. The result will be that

$$
\bar{d}\left(\left\{\left.f^{-i} \xi\right|_{A}\right\}_{i=1+m}^{n+m},\left\{f^{-i} \xi\right\}_{i=1+m}^{n+m}\right)<c \varepsilon
$$

where $c$ depends only on our space $G S$. That is, $f^{-m} \xi$ is VWB.
A rough outline of the argument for the construction of $\xi$ and $\psi$ as in Proposition 4.9 is as follows.
(1) Let $\xi$ be a partition of $G S$ with a small amount of measure concentrated near the boundaries of its sets.
(2) Define an auxiliary "almost partition" of $G S$ into dynamical rectangles, where on each rectangle, $\mu$ is equivalent or absolutely continuous with respect to a product measure supported on the local unstable and center-stable manifolds.
(3) Show that we can choose $M$ so that for $m_{2} \geq m_{1} \geq M$, "most" partition elements $A \in \bigvee_{m_{1}}^{m_{2}} f^{i} \xi$ have a subset of relatively large measure $S_{A} \subset A$ which stretches completely across the rectangle.
(4) Given a set $S_{A}$ which "stretches across" a rectangle $R$ in the unstable direction, show that we can define a map $\psi_{R}: S_{A} \cap R \rightarrow R$ which is product measure-preserving, and maps points along their local center stable sets.
(5) After identifying a small-measure collection of 'bad' atoms from $\bigvee_{k=N_{0}}^{N_{1}} f^{k} \xi$, given a 'good' atom $A$, glue together $\psi: A \rightarrow G S$ from the $\psi_{R}$ 's, and then show that it meets the necessary criteria from Lemma 4.9.
Of these steps, only (2), (3), and (5) require hyperbolicity of some kind. Step (1) is an entirely general construction for which we quote a Lemma from OW98. The main work for Step (2) was accomplished in Section 3.3. We discuss the first two steps in Section 4.3 below. In Section 4.4 we give a precise definition
and prove the desired 'layerwise' intersection property of Step (3). The argument is completed with Steps (4) and (5) in Sections 4.5 and 4.6, primarily by recalling the argument of OW73.

Once the argument above is complete, we finish the proof of Theorem B by invoking a pair of theorems from OW73:

Theorem 4.10 (OW73], Theorems A and B).
(A) If $\xi$ is a $V W B$ partition of the system $(X, \mu, f)$, then $\left(X, \bigvee_{i=-\infty}^{\infty} f^{i} \xi, \mu, f\right)$ is Bernoulli.
(B) If $\mathcal{A}_{1} \subseteq \mathcal{A}_{2} \subseteq \cdots$ is an increasing sequence of $f$-invariant $\sigma$-algebras with $\mathcal{B}=\bigvee_{n=1}^{\infty} \mathcal{A}_{n}$, and each $\left(X, \mathcal{A}_{n}, \mu, f\right)$ is Bernoulli, then $(X, \mathcal{B}, \mu, f)$ is Bernoulli.

Proof of Theorem $B$. We apply Theorem 4.10 to $(G S, \mathcal{B}, \mu, f)$ with $f=g_{1}$ and $\mathcal{B}$ the Borel $\sigma$-algebra. Our argument in the following sections will show that for any partition $\xi$ with a small amount of measure concentrated near its boundary, $f^{-m} \xi$ is VWB. Proposition 4.11 below ensures that such $\xi$ exist. Indeed, it ensures that such $\xi$ with arbitrarily small diameter can be constructed. Applying this Proposition inductively, we can find a sequence $\xi_{1} \subseteq \xi_{2} \subseteq \cdots$ with diameters going to zero. $\bigvee_{n=1}^{\infty} \xi_{n}$ certainly generates $\mathcal{B}$.

Applying 4.10(A), we find that $\left(G S, \bigvee_{i=-\infty}^{\infty} f^{i} \xi_{n}, \mu, f\right)$ is Bernoulli. Since $\mathcal{A}_{n}:=\bigvee_{i=-\infty}^{\infty} f^{i} \xi_{n}$ are $f$ invariant, and $\mathcal{B}=\bigvee_{n=1}^{\infty} \mathcal{A}_{n}$, by Theorem 4.10(B), $(G S, \mathcal{B}, \mu, f)$ is Bernoulli.
4.3. Steps 1 \& 2: Good partitions. The first step toward using Proposition 4.9 is to note the existence of a partition of $G S$ such that the measure near its boundary is controlled. This is given by the following general fact.

Proposition 4.11 (See OW98, Lemma 4.1 and following remarks). Let $(X, \mu)$ be a compact metric space, $\mu$ a probability measure. For all $\epsilon>0$, there exists a partition $\xi$ of diameter less than $\epsilon$ such that there exists $C>0$ such that for all $r>0$,

$$
\mu(B(\partial \xi, r)) \leq C r
$$

where $B(\partial \xi, r)$ is the $r$-neighborhood of the union of the boundaries of the sets in $\xi$.
The second step is to find an "almost partition" of $G S$ using rectangular subsets of the flow boxes constructed in Section 3.3. We will denote these rectangles by $R$ or $R_{i}$. This matches the presentation of CH96, for example, but these $R$ 's should not be confused with the rectangle transversals considered earlier.

Towards that end, note that we may assume (by decreasing $\varepsilon$ and increasing $n$ as necessary) that the bracket structure provided by $[-,-]$ is defined at scale $\bar{\delta}$ which is at least twice the diameter of any such $\mathcal{B}\left(n_{0}, \varepsilon\right)$.

There are product coordinates on $\mathcal{B}(n, \varepsilon)$, defined as follows. Let $z \in \mathcal{B}(n, \varepsilon)$. We assign it coordinates

$$
z \mapsto\left(z^{u}, z^{s}, z^{c}\right) \in W^{u}(x, \varepsilon) \times W^{s}(x, \varepsilon) \times\left[-\frac{1}{n}, \frac{1}{n}\right]
$$

where

$$
\begin{aligned}
z^{u}= & W^{u}(x, \varepsilon) \cap W^{c s}(z, \bar{\delta}) \\
z^{s}= & W^{s}(x, \varepsilon) \cap W^{c u}(z, \bar{\delta}) \\
& g_{z^{c}}\left\langle z^{u}, z^{s}\right\rangle=z
\end{aligned}
$$

(See Figure 3)
We note that topologically, $W^{u}(x, \varepsilon)$ and $W^{s}(x, \varepsilon)$ are intervals - they can be identified via projecting to the forward (resp. backward) endpoints at infinity with subsets of the circle $\partial_{\infty} \tilde{S}$.

Definition 4.12. A subset $A \subset \mathcal{B}(n, \varepsilon)$ is a rectangle if there are intervals (open, closed, or half open) $I^{u} \subset W^{u}(x, \varepsilon), I^{s} \subset W^{s}(x, \varepsilon)$, and $I^{c} \subset\left[-\frac{1}{n}, \frac{1}{n}\right]$ such that

$$
A=\left\{z: z^{u} \in I^{u}, z^{s} \in I^{s}, z^{c} \in I^{c}\right\}
$$

or, equivalently, under the map to the coordinates,

$$
A \mapsto I^{u} \times I^{s} \times I^{c}
$$



Figure 3. The product coordinates on a flow box.
Suppose that $B_{1}:=\mathcal{B}\left(n_{1}, \varepsilon_{1}\right)$ centered at $x_{1}$ and $B_{2}:=\mathcal{B}\left(n_{2}, \varepsilon_{2}\right)$ centered at $x_{2}$ intersect. Since the diameters of $B_{1}$ and $B_{2}$ are less than half the scale on which the bracket $[-,-]$ is defined, we can relate the coordinates for $B_{1}$ and $B_{2}$.

Let $z \in B_{1} \cap B_{2}$ with coordinates $\left(z_{1}^{u}, z_{1}^{s}, z_{1}^{c}\right)$ for $B_{1}$ and $\left(z_{2}^{u}, z_{2}^{s}, z_{2}^{c}\right)$ for $B_{2}$.

- Unstable coordinates: $z_{i}^{u}$ both lie on $W^{c s}(z, \bar{\delta})$. Hence,

$$
z_{1}^{u}=W^{u}\left(x_{1}, \varepsilon_{1}\right) \cap W^{c s}\left(z_{2}^{u}, \bar{\delta}\right) .
$$

- Stable coordinates: $z_{i}^{s}$ both lie on $W^{c u}(z, \bar{\delta})$. Hence,

$$
z_{1}^{s}=W^{s}\left(x_{1}, \varepsilon_{1}\right) \cap W^{c u}\left(z_{2}^{s}, \bar{\delta}\right) .
$$

- Flow coordinates: $z_{1}^{c}-z_{2}^{c}=t^{*}$ where $g_{t^{*}}\left\langle z_{1}^{u}, z_{1}^{s}\right\rangle=\left\langle z_{2}^{u}, z_{2}^{s}\right\rangle$. Note that $t^{*}$ is the distance along the flow through $z$ between $R_{1}\left(\varepsilon_{1}\right)$ and $R_{2}\left(\varepsilon_{2}\right)$. This distance is constant in $z$ as $R_{i}\left(\varepsilon_{i}\right)$ are connected and foliated by stable and unstable leaves, and the flow takes (un)stable leaves to (un)stable leaves. Hence,

$$
z_{1}^{c}=t^{*}+z_{2}^{c} .
$$

The transformation of the flow coordinate obviously maps intervals to intervals. The same is true for the other foliations. To see that these maps preserve the ordering of points on the (un)stable leaves they map between, one can either draw a geometric picture of the maps in $\tilde{S}$ or note that these maps are given by holonomies along the leaves of the center (un)stable foliation. Since distinct leaves cannot intersect, ordering must be preserved.

The work above allows us to prove the following.
Lemma 4.13. If $A$ is a rectangle in $B_{2}$, then $A \cap B_{1}$ is a rectangle in $B_{1}$.
Proof. Suppose that $A$ has coordinates $I_{2}^{u} \times I_{2}^{s} \times I_{2}^{c}$ in $B_{2}$. Then in $B_{1}$ it has the coordinates $I_{1}^{u} \times I_{1}^{s} \times I_{1}^{c}$ with

$$
\begin{gathered}
I_{1}^{u}=\left\{z_{1}^{u}: z_{2}^{u} \in I_{2}^{u}\right\}=\left\{W^{u}\left(x_{1}, \varepsilon_{1}\right) \cap W^{c s}\left(z_{2}^{u}, \bar{\delta}\right): z_{2}^{u} \in I_{2}^{u}\right\} \\
I_{1}^{s}=\left\{z_{1}^{s}: z_{2}^{s} \in I_{2}^{s}\right\}=\left\{W^{s}\left(x_{1}, \varepsilon_{1}\right) \cap W^{c u}\left(z_{2}^{s}, \bar{\delta}\right): z_{2}^{s} \in I_{2}^{s}\right\} \\
I_{1}^{c}=\left\{z_{1}^{c}: z_{s}^{c} \in I_{2}^{c}\right\} .
\end{gathered}
$$

Since the maps $z_{2}^{*} \rightarrow z_{1}^{*}$ map intervals to intervals, $A$ is a rectangle in $B_{1}$.

Proposition 4.14. If $A_{1}$ and $A_{2}$ are rectangles in $B_{1}$ and $B_{2}$, then $A_{1} \cap A_{2}$ is a rectangle in $B_{1}$ and $A_{1} \backslash A_{2}$ is a finite union of rectangles in $B_{1}$.
Proof. By Lemma 4.13, $A_{2} \cap B_{1}$ is a rectangle in $B_{1}$. $A_{1} \cap A_{2}=A_{1} \cap\left(A_{2} \cap B_{1}\right)$ and $A_{1} \backslash A_{2}=A_{1} \backslash\left(A_{2} \cap B_{1}\right)$ so we can without loss of generality assume $A_{1}$ and $A_{2}$ are both rectangles in the same flow box $B_{1}$.

In the stable/unstable/flow coordinates $A_{1}$ and $A_{2}$ are rectangles in $W^{u}\left(x_{1}, \varepsilon_{1}\right) \times W^{s}\left(x_{2}, \varepsilon_{2}\right) \times\left[-\frac{1}{n_{1}}, \frac{1}{n_{1}}\right]$, itself a rectangle in $\mathbb{R}^{3}$ after identifying these (un)stable leaves with intervals. The result now follows from the analogous fact about rectangles in $\mathbb{R}^{3}$.

Lemma 4.15. For any rectangle $A$ with $\mu(A)>0$ in a flow box $\mathcal{B}\left(n_{0}, \varepsilon\right)$ on which $\mu$ has local product structure, the conditional measure $\left.\mu\right|_{A}$ has local product structure.
Proof. As noted at the start of Section 3.3.3. local product structure can be deduced from the behavior of conditional measures under holonomy maps described in Proposition 3.30 and Corollary 3.31. When $\mu$ is conditioned on $A$ (using the fact that $\mu(A)>0$ ), the resulting conditional measures are renormalized versions of the measures for the full flow box and the holonomy maps are the same. Hence, these results hold for $A$, and $\left.\mu\right|_{A}$ has local product structure.

The conditions on the partition we want are described in the following definition.
Definition 4.16. For a given $\delta>0$, a partition $\pi=\left\{R_{0}, R_{1}, \ldots, R_{k}\right\}$ is a good almost partition of $G S$ into rectangles if

- Each $R_{i}$ for $i=1, \ldots, k$ is a rectangle as in Definition 4.12 with diameter $<\delta / C$ where $C$ is from Proposition 4.11.
- $R_{0}$ is some measurable set with $\mu\left(R_{0}\right)<\delta$.
- If $x, y \in R_{i}(i=1, \ldots, k)$ are on the same local unstable, then there is a curve in that local unstable of length $<8 \operatorname{diam}\left(R_{i}\right)<8 \delta / C$ joining them.
- For every $R_{i}(i=1, \ldots, k)$

$$
\left|\frac{\mu_{R_{i}}^{p}\left(R_{i}\right)}{\mu\left(R_{i}\right)}-1\right|<\delta
$$

where $\mu_{R_{i}}^{p}$ is the local product structure measure on $R_{i}$. Each $R_{i}(i=1, \ldots, k)$ contains a subset $G_{i}$ with $\mu\left(G_{i}\right)>(1-\delta) \mu\left(R_{i}\right)$ such that at all points $x \in G_{i}$,

$$
\left|\frac{d \mu_{R_{i}}^{p}}{d \mu}(x)-1\right|<\delta
$$

- If $x, y \in R_{i}(i=1, \ldots, k)$ are on the same local stable leaf, the distance between them contracts exponentially fast under the geodesic flow.

Proposition 4.17. Given any $\delta>0$ there exists a good almost partition of $G S$ into dynamical rectangles.
Proof. In Corollary 3.31 we proved that almost every $\gamma \in G S$ - specifically, any $\gamma$ with $\lambda\left(g_{t} \gamma\right)>0$ for all $t \in \mathbb{R}$ - belongs to a flow box $\mathcal{B}(n, \varepsilon)$ on which $\mu$ has local product structure. In Lemma 4.15 we showed that on any rectangle $R$ with $\mu(R)>0$ in such a flow box, $\mu$ also has local product structure. We will create our good almost partition from such rectangles. They are clearly measurable, and hence $R_{0}$ will be measurable.

First, note that we can choose the rectangles $R$ with diameter smaller than $\delta / C$ where $C$ is from Proposition 4.11.

Second, it is clear from the construction of the flow boxes (see Definitions 2.30 and 2.33 , and take $n$ sufficiently large) that the unstable segments in each flow box, and hence each rectangle within it are of length $<8 \operatorname{diam}(R)$. Also by construction (see Lemma 2.27) if $x, y \in R$ are on the same stable leaf, then $x(t)=y(t)$ for all $t \geq 0$. Lemma 2.12 gives the desired exponential contraction.

We know from Lemma 4.15 that $\frac{d \mu}{d \mu_{R}^{p}} \in\left[\bar{K}^{-1}, \bar{K}\right]$ on each such rectangle. Hence, $\mu_{R}^{p} \ll \mu$ as well. $\frac{d \mu_{R}^{p}}{d \mu}$ is measurable, so at almost every point, on a sufficiently small neighborhood, it is nearly constant off a small measure set. Therefore, shrinking $R$ as necessary, and rescaling $\mu_{R}^{p}$ as necessary, we can ensure that $\left|\frac{d \mu_{R}^{p}}{d \mu}-1\right|<\delta$ on a set $G \subset R$ with $\mu(G)>(1-\delta) \mu(R)$. Combining this with our bounds on $\frac{d \mu}{d \mu_{R}^{p}}$ we can show that $\left|\frac{\mu_{R}^{p}(R)}{\mu(R)}-1\right|<\delta$.

Therefore, around almost every point in $G S$ we can construct a rectangle answering the requirements of Definition 4.16 of arbitrarily small diameter. Let $E$ be the full measure set of such points. The fact that we can make our rectangles of arbitrarily small diameter and the results of Proposition 4.14, show that these rectangles form a generating semi-ring for the restriction of the $\sigma$-algebra of measurable sets to $E$. Therefore, every measurable set in this $\sigma$-algebra, including $E$ itself, can be approximated up to measure $<\delta$ by a finite, disjoint, union of our rectangles. These rectangles, together with their complement of measure $<\delta$, is the partition $\pi$.
4.4. Step 3: Layerwise intersection. The next step is to prove (Proposition 4.22, below) that given a partition $\xi$ as provided in Proposition 4.11 and a collection of small rectangles, there is a time $M$ such that, after pushing $\xi$ forward by times $\geq M$, almost all of its subsets intersect each rectangle in our collection 'layerwise.'

Definition 4.18. A measurable set $S$ intersects a rectangle $R$ layerwise if for all $x \in S \cap R, V_{x}^{u, S} \supseteq V_{x}^{u, R}$.
First, we record a geometric lemma.
Proposition 4.19. Let $\epsilon>0$ and $t_{0} \geq 0$ be given. Then there exists a set $E \subset G X$ (depending on $\varepsilon$ and $t_{0}$ ), an $\eta>0$ (depending only on $\varepsilon$ ), and $A, B \geq t_{0}$ (depending on $\varepsilon$ and $t_{0}$ ) such that:

- $\mu(E)<\frac{\varepsilon^{2}}{2}$
- for each $\gamma \in E^{c}$, there exists a $t_{\gamma}^{\prime} \in[A, B]$ such that $\left|\theta\left(\gamma, t_{\gamma}^{\prime}\right)-\pi\right| \geq c$ for some $c>0$.

Proof. Recall that Sing $=\bigcap_{t \in \mathbb{R}} g_{t}(\{\lambda=0\})$. (Here, $\{\lambda=0\}$ is shorthand for $\{\gamma \in G X: \lambda(\gamma)=0\}$.) Thus,

$$
\text { Sing }=\bigcap_{c^{\prime}>0} \bigcap_{t \in \mathbb{R}} g_{t}\left(\left\{\lambda<c^{\prime}\right\}\right)
$$

By Corollary 3.8, $\mu(\operatorname{Sing})=0$ for our equilibrium states, so given $\varepsilon>0$, there exists $c^{*}>0$ such that

$$
\mu\left(\bigcap_{t \in \mathbb{R}} g_{t}\left(\left\{\lambda<c^{*}\right\}\right)\right)<\frac{\varepsilon^{2}}{4} .
$$

Now,

$$
\bigcap_{t \in \mathbb{R}} g_{t}\left(\left\{\lambda<c^{*}\right\}\right)=\bigcap_{n \in \mathbb{N}} \bigcap_{t \in[-n, n]} g_{t}\left(\left\{\lambda<c^{*}\right\}\right) .
$$

Therefore, there exists $N_{1} \in \mathbb{N}$ such that

$$
\mu\left(\bigcap_{t \in\left[-N_{1}, N_{1}\right]} g_{t}\left(\left\{\lambda<c^{*}\right\}\right)\right)<\frac{\varepsilon^{2}}{2} .
$$

Let

$$
F=\bigcap_{t \in\left[-N_{1}, N_{1}\right]} g_{t}\left(\left\{\lambda<c^{*}\right\}\right) .
$$

Then

$$
\left.F^{c}=\left\{\gamma \in G X: \exists t \text { with }|t| \leq N_{1} \text { such that } \lambda\left(g_{t} \gamma\right) \geq c^{*}\right)\right\}
$$

and $\mu\left(F^{c}\right) \geq 1-\frac{\varepsilon^{2}}{2}$.
Let $E=g_{-N_{1}-t_{0}-\frac{\theta_{0}}{2 \eta}} F ; E^{c}=g_{-N_{1}-t_{0}-\frac{\theta_{0}}{2 \eta}} F^{c}$. We still have $\mu\left(E^{c}\right) \geq 1-\frac{\varepsilon^{2}}{2}$. If $\gamma \in E^{c}$, then there exists a time $t_{\gamma} \in\left[t_{0}+\frac{\theta_{0}}{2 c^{*}}, t_{0}+\frac{\theta_{0}}{2 c^{*}}+2 N_{1}\right]$ such that $\lambda\left(g_{t} \gamma\right) \geq c^{*}$. Let $c=\frac{c^{*}}{2 s}$. Then for each $\gamma \in E^{c}$, there exists a $t_{\gamma}^{\prime} \in\left[t_{0}, t_{0}+2 N_{1}\right]=:[A, B]$ such that $\left|\theta\left(\gamma, t_{\gamma}^{\prime}\right)-\pi\right| \geq c$.

Definition 4.20. Let $E$ be as provided by Proposition 4.19. Given a collection $\mathcal{R}$ of rectangles, we decompose it as $\mathcal{R}=\mathcal{R}_{E} \cup \mathcal{R}_{E^{c}}$, where $\mathcal{R}_{E}=\{R \in \mathcal{R}: R \subset E\}$ and $\mathcal{R}_{E^{c}}=\left\{R \in \mathcal{R}: R \cap E^{c} \neq \emptyset\right\}$.

Directly from Proposition 4.19 we have

Corollary 4.21. For any collection $\mathcal{R}$ of rectangles,

$$
\mu\left(\bigcup_{R \in \mathcal{R}_{E}} R\right)<\frac{\varepsilon^{2}}{2}
$$

In all that follows, $\xi=\left\{F_{j}\right\}$ is a measurable partition of $G S . B(\partial \xi, \varepsilon)$ is the $\varepsilon$-neighborhood of the union of the boundaries of the sets in $\xi$. We assume, using Proposition 4.11, that there exists $C>0$ so that for all $r>0, \mu(B(\partial \xi, r))<C r$.

Fix $\varepsilon>0$ and set $t_{0}=2$. Let $c>0, E \subset G S$, and $A, B \geq t_{0}$ be as described in Proposition 4.19 (for our choice of $\epsilon$ and $t_{0}$ ). Recall that each $\gamma \in E^{c}$ has a time $t_{\gamma}^{\prime} \in[A, B]$ associated to it for which $\theta\left(\gamma, t_{\gamma}^{\prime}\right) \geq \eta$.

Given $c$ and $B$, let $\delta>0$ be as given in Lemma 4.2. By Lemma 4.2 if $q \in E^{c}$ and $x, y \in B(q, \delta)$ with $y \in V_{x}^{u, B(q, \delta)}$, then $x$ and $y$ agree over times in $\left(-\infty, t_{q}^{\prime}\right]$. Therefore, Lemma 2.11 applies for such $x, y$.

We are now ready for the main proposition of Step 4:
Proposition 4.22. Let $\epsilon>0$ be given. Let $\mathcal{R}$ be a finite, pairwise disjoint collection of rectangles with diameters $<\delta$.

If $\xi$ is a partition as described above, then there exists $M$ (depending on the $C$ mentioned in the description of $\xi$ as well as on $\varepsilon$ ) such that for all $m_{2} \geq m_{1} \geq M$ and for $\varepsilon$-almost every $F \in \bigvee_{k=m_{1}}^{m_{2}} f^{k} \xi$ there is a set $S_{F} \subset F$ with $\left.\mu\right|_{F}\left(S_{F}\right)>1-\varepsilon$ which intersects each $R \in \mathcal{R}$ layerwise.

Proof. Let $\varepsilon>0$ be given; set $t_{0}=2$. Let $E$ be as provided by Proposition4.19 it depends on $\epsilon$ and $t_{0}$.
Let $\mathcal{R}$ be a collection of rectangles with diameters $<\delta$. Decompose $\mathcal{R}=\mathcal{R}_{E} \cup \mathcal{R}_{E^{c}}$ as per Definition 4.20 Let $q_{R} \in R \in \mathcal{R}_{E^{c}}$ for all $R \in \mathcal{R}_{E^{c}}$.

Recall that $\xi$ is a partition such that $\mu\left(B\left(\partial \xi, r^{\prime}\right)\right)<C r^{\prime}$ for all $r^{\prime}>0$. Let $M$ be so large that $C \delta \sum_{k=M}^{\infty} e^{-k}<\frac{\varepsilon^{2}}{2}$.

Let $F \in \bigvee_{k=m_{1}}^{m_{2}} f^{k} \xi$. Then if $\xi=\left\{F_{j}\right\}$,

$$
F=\bigcap_{k=m_{1}}^{m_{2}} f^{k} F_{j_{k}} .
$$

A point $x$ is in $F$ if and only if for all $k \in\left[m_{1}, m_{2}\right], f^{-k} x \in F_{j_{k}}$.
Let $S_{1}=F \backslash\left(\cup_{R \in \mathcal{R}} R\right)$. Let

$$
S_{2}=\bigcup_{R \in \mathcal{R}_{E^{c}}}\left\{x \in F \cap R: V_{x}^{u, R} \subseteq V_{x}^{u, F}\right\} .
$$

Let $S_{F}=S_{1} \cup S_{2}$. Our goal is to prove that $\left.\mu\right|_{F}\left(S_{F}^{c}\right) \leq \varepsilon$.
Suppose that $x \notin S_{F}$. Then $x$ belongs to some rectangle $R \in \mathcal{R}$. If that rectangle is in $\mathcal{R}_{E}$, then $x \in S_{F}^{c} \cap\left(\cup_{R \in \mathcal{R}_{E}} R\right)$. By Corollary $4.21, \mu\left(S_{F}^{c} \cap\left(\cup_{R \in \mathcal{R}_{E}} R\right)\right)<\frac{\varepsilon^{2}}{2}$.

If, on the other hand, that rectangle is in $R_{E^{c}}$, then there exists some $y \in V_{x}^{u, R}$ such that $y \notin F$. Since $y \notin F$, there exists some $k \in\left[m_{1}, m_{2}\right]$ such that $y \notin f^{k} F_{j_{k}}$. As they are both in $R, d_{G X}(x, y)<\delta$. Since $y \in V_{x}^{u, R}$ and $R$ is a good rectangle, by Lemma 2.11.

$$
d_{G X}\left(g_{-k} x, g_{-k} y\right)<\delta e^{-k}
$$

Therefore, $g_{-k} x \in B\left(\partial \xi, \delta e^{-k}\right)$.
Let

$$
X_{k}=g_{k} B\left(\partial \xi, \delta e^{-k}\right)
$$

By the choice of $\xi, \mu\left(X_{k}\right)<C \delta e^{-k}$.
We have shown that for each $F \in \bigvee_{k=m_{1}}^{m_{2}} f^{k} \xi$,

$$
F \backslash S_{F} \subset\left(\bigcup_{R \in \mathcal{R}_{E}} R\right) \cup \bigcup_{k=m_{1}}^{m_{2}} X_{k}
$$

Therefore, letting $Z=\bigcup_{F \in \xi}\left(F \backslash S_{F}\right)$,

$$
\mu(Z) \leq \mu\left(\bigcup_{R \in \mathcal{R}_{E}} R\right)+\sum_{k=m_{1}}^{m_{2}} \mu\left(X_{k}\right)<\frac{\varepsilon^{2}}{2}+C \delta \sum_{k=M}^{\infty} e^{-k}<\frac{\varepsilon^{2}}{2}+\frac{\varepsilon^{2}}{2}=\varepsilon^{2}
$$

Writing $\bigvee_{k=m_{1}}^{m_{2}} f^{k} \xi=\left\{F_{j}: j \in J\right\}$, let $J^{\prime}=\left\{j \in J:\left.\mu\right|_{F_{j}}\left(F_{j} \backslash S_{F_{j}}\right) \geq \varepsilon\right\}$. Suppose that

$$
\mu\left(\bigcup_{j \in J^{\prime}} F_{j}\right)=\sum_{j \in J^{\prime}} \mu\left(F_{j}\right) \geq \varepsilon .
$$

Then

$$
\begin{aligned}
\mu(Z) & =\sum_{j \in J} \mu\left(Z \cap F_{j}\right)=\left.\sum_{j \in J} \mu\left(F_{j}\right) \mu\right|_{F_{j}}(Z) \\
& \geq\left.\sum_{j \in J^{\prime}} \mu\left(F_{j}\right) \mu\right|_{F_{j}}(Z) \geq \sum_{j \in J^{\prime}} \mu\left(F_{j}\right) \varepsilon \geq \varepsilon^{2} .
\end{aligned}
$$

This contradicts the fact that $\mu(Z)<\varepsilon^{2}$. Therefore, for $\varepsilon$-almost every $F \in \bigvee_{k=m_{1}}^{m_{2}} f^{k} \xi$, namely for $\left\{F_{j}: j \notin J^{\prime}\right\},\left.\mu\right|_{F}\left(F \backslash S_{F}\right)<\varepsilon$. The sets $S_{F}$ are the ones we want.
4.5. Step 4: Local definition of $\psi$. We now turn to the construction of $\psi$ satisfying the conditions of Lemma 4.9. At this point, the argument entirely follows OW73, §3] or CH96, §6.2], so we summarize the main points here, clarifying how their argument unfolds in our setting.

We begin with Step (4), defining $\psi_{i}: S_{A} \cap R_{i} \rightarrow R_{i}$ for each rectangle from our auxiliary partition for which $\mu\left(S_{A} \cap R\right)>0$. These $\psi_{i}$ will be combined to form $\psi$ in Step (5).

Consider those rectangles $R_{i}$ in $\pi(i=1, \ldots, n)$, such that $S_{A} \cap R_{i}$ has positive measure, where $S_{A}$ is given by Proposition 4.22. Recall that for any $y \in S_{A} \cap R_{i}, V_{x_{i}}^{u, R_{i}} \subset V_{x_{i}}^{u, A}$; that is, the connected component of $y$ 's unstable leaf in $A$ stretches fully across $R_{i}$.

For each such $i$, pick a point $x_{i} \in S_{A} \cap R_{i}$. Consider $V_{x_{i}}^{c s, S_{A} \cap R_{i}}$ and $V_{x_{i}}^{c s, R_{i}}$. These can each be equipped with factor measures $\left.\mu_{x_{i}}^{c s}\right|_{V_{i}} ^{c s, S_{A} \cap R_{i}}$ and $\left.\mu_{x_{i}}^{c s}\right|_{V_{x_{i}}} ^{c s, R_{i}}$ from the local product structure on $R_{i}$. Each of these is a non-atomic, Lebesgue, probability measure space (see the Remark just before Proposition 3.28, so there is a bijective, measure-preserving map

$$
\psi_{i}:\left(V_{x_{i}}^{c s, S_{A} \cap R_{i}},\left.\mu_{x_{i}}^{c s}\right|_{V_{x_{i}}^{c s, S_{A} \cap R_{i}}}\right) \rightarrow\left(V_{x_{i}}^{c s, R_{i}},\left.\mu_{x_{i}}^{c s}\right|_{V_{x_{i}}^{c s, R_{i}}}\right) .
$$

Define $\psi$ on $V_{x_{i}}^{c s, S_{A} \cap R_{i}}$ by $\psi_{i}$.
Let $y \in S_{A} \cap R_{i}$. Define $\psi_{i}(y)$ with this composition:

$$
y \mapsto W_{y}^{u} \cap W_{x_{i}}^{c s} \mapsto \psi_{i}\left(W_{y}^{u} \cap W_{x_{i}}^{c s}\right) \mapsto W_{\psi_{i}\left(W_{y}^{u} \cap W_{x_{i}}^{c s}\right)}^{u} \cap W_{y}^{c s} .
$$

That is, map $y$ along the strong unstable to the weak stable of $x_{i}$. Since $R_{i}$ is a rectangle, $W_{y}^{u} \cap W_{x_{i}}^{c s}$ is still in $R_{i}$, and since $y$ is in $S_{A}, W_{y}^{u} \cap W_{x_{i}}^{c s}$ is still in $S_{A}$. Hence, $\psi_{i}\left(W_{y}^{u} \cap W_{x_{i}}^{c s}\right)$ is defined. Then, using the rectangular structure of $R_{i}$ once again, we push $\psi_{i}\left(W_{y}^{u} \cap W_{x_{i}}^{c s}\right)$ back to the weak stable of $y$. (See Figure 4)

The construction of $\psi_{i}$ involves translations along the strong unstable factor of the product composed with a measure preserving map in the weak stable leaf. Therefore, $\psi_{i}:\left(S_{A} \cap R_{i}, \mu_{R_{i}}^{p} \mid S_{A}\right) \rightarrow\left(R_{i}, \mu_{R_{i}}^{p}\right)$ is measure-preserving for the (correctly conditioned) product measures. This completes Step (4).
4.6. Step 5: Completing the argument. Turning to Step (5), recall that by Step (1) (i.e., Proposition 4.11) $\xi$ is a partition of $(G S, \mu)$ with $\mu(B(\partial \alpha, r))<C r$ for any $r>0$. Let $\varepsilon$ be given, and set $\delta=\varepsilon^{4}$. Recall the partition $\pi=\left\{R_{0}, R_{1}, \ldots, R_{k}\right\}$ where $R_{i}(i=1, \ldots, k)$ are the rectangles provided by Step (2) and $\mu\left(R_{0}\right)<\delta$. For $i=1, \ldots, k$, let $G_{i} \subset R_{i}$ be the set described in Step (2).

By the $K$-property for $\left(G S, \mu, g_{t}\right)\left(\left[\overline{\mathrm{CCE}^{+} 23}\right.\right.$, Theorem A] $)$, there exists an integer $m$ such that, setting $N=2 m$, for any integers $N_{1}>N_{0}>N, \delta$-a.e. atom $A$ of $\bigvee_{N_{0}-m}^{N_{1}-m} f^{i} \alpha$ satisfies, for every $R \in \pi$,

$$
\begin{equation*}
\left|\frac{\left.\mu\right|_{A}(R)}{\mu(R)}-1\right|<\delta . \tag{3}
\end{equation*}
$$

By Proposition 4.22 (with $\epsilon$ in that Proposition taken to be our current $\delta$ ), we can also take $N=2 m$ so large that whenever $N_{1}>N_{0}>N$, $\delta$-a.e. atom $A$ of $\bigvee_{N_{0}-m}^{N_{1}-m} f^{i} \alpha$ contains a set $S_{A} \subset A$ with $\left.\mu\right|_{A}\left(S_{A}\right)>1-\delta$ intersecting $R$ layerwise.

Let $m, N, N_{0}, N_{1}$ as above and $n>0$ be given. Let $\omega=\bigvee_{i=N_{0}-m}^{N_{1}-m} f^{i} \alpha$. We want to prove, for $c \varepsilon$-a.e. atom $A$ of $\omega$ that

$$
\bar{d}\left(\left\{f^{-i} \alpha\right\}_{i=1+m}^{n+m}, \underset{34}{\left.\left\{\left.f^{-i} \alpha\right|_{A}\right\}_{i=1+m}^{n+m}\right)<c \varepsilon}\right.
$$



Figure 4. The definition of $\psi_{i} . y^{\prime}=W_{y}^{u} \cap W_{x_{i}}^{c s}, y^{\prime \prime}=\psi_{i}\left(W_{y}^{u} \cap W_{x_{i}}^{c s}\right)$. $\psi_{i}$ spreads out the dark blue region to the light blue region in a (conditional) measure-preserving way.
holds.
At this point, we can state the definition of $\psi$. For $A \in \omega$, if $y \in S_{A} \cap R_{i}(i=1, \ldots, n)$ where $\mu\left(S_{A} \cap R_{i}\right)>0$, set $\psi(y)=\psi_{i}(y)$. For all other $y$, set $\psi(y)=y$.

It remains to verify the conditions of Lemma 4.9 for $c \varepsilon$-almost every $A \in \bigvee_{k=N_{0}}^{N_{1}} f^{k} \xi$. We begin by identifying the 'bad' atoms in $\bigvee_{k=N_{0}}^{N_{1}} f^{k} \xi$, and ensuring that their total measure is less than $c \varepsilon$.

Let

$$
\hat{F}_{1}=\bigcup_{A \in \omega: A \text { fails (3) }} A .
$$

By our choice of $N, \mu\left(\hat{F}_{1}\right)<\delta$.
Let $F_{2}=\bigcup_{i=1}^{k}\left(R_{i} \backslash G_{i}\right)$. By Step (2), $\mu\left(F_{2}\right)<\delta$. In addition, a brief computation using facts from Step (2) implies

$$
\begin{equation*}
\sum_{i=1}^{k} \mu_{R_{i}}^{p}\left(F_{2} \cap R_{i}\right)<c \delta \tag{4}
\end{equation*}
$$

for some universal $c$. Let

$$
\hat{F}_{2}=\bigcup_{A \in \omega: \text { either }\left.\mu\right|_{A}\left(F_{2}\right)>\delta^{1 / 2} \text { or } \sum_{i=1}^{k} \frac{\mu_{R_{i}}^{p}\left(A \cap F_{2}\right)}{\mu(A)}>\delta^{1 / 2}} A .
$$

Some computation shows that $\mu\left(\hat{F}_{2}\right)<c \delta^{1 / 2}$ for some $c$. We provide part of the argument here, as it is illustrative of a general strategy for bounding the measures $\mu\left(\hat{F}_{i}\right)$ throughout (see also CH96). Recall that $\mu\left(F_{2}\right)<\delta$. Then $\left.\mu\right|_{A}\left(F_{2}\right):=\frac{\mu\left(F_{2} \cap A\right)}{\mu(A)}>\delta^{1 / 2}$ implies that for such $A, \mu(A)<\delta^{-1 / 2} \mu\left(F_{2} \cap A\right)$. Hence,

$$
\mu\left(\bigcup_{A \in \omega:\left.\mu\right|_{A}\left(F_{2}\right)>\delta^{1 / 2}} A\right)<\sum_{\text {all } A \in \omega} \delta^{-1 / 2} \mu\left(F_{2} \cap A\right)=\delta^{-1 / 2} \mu\left(F_{2}\right)<\delta^{1 / 2} .
$$

A similar computation using (4) bounds the other part of the union for $\hat{F}_{2}$.
Let

$$
F_{3}=\left\{x \in G S \backslash R_{0}: x \notin S_{A}\right\}
$$

where $S_{A}$ is provided by applying Proposition 4.22 to $A$ and the collection $\mathcal{R}=\left\{R_{1}, \ldots, R_{k}\right\}$. Let

$$
\hat{F}_{3}=\bigcup_{A \in \omega:\left.\mu\right|_{A}\left(F_{3}\right)>\delta^{1 / 2}} A
$$

By similar arguments, using the fact that $\left.\mu\right|_{A}\left(S_{A}\right)>1-\delta$, we find that $\mu\left(\hat{F}_{3}\right)<c \delta^{1 / 2}$ for some $c$.
Finally, let

$$
F_{4}=\left\{x \in G S \backslash R_{0}: \exists y \in V_{x}^{c s, \pi(x)} \text { such that } h(x, y)>\delta^{1 / 2}\right\}
$$

If $x \in F_{4}$, then $x$ and $y$, two points on the same weak stable which are $<\operatorname{diam} \pi(x)<\delta / C$ apart are, with relative frequency $>\delta^{1 / 2}$ over the integers $1+m \leq i \leq n+m$, in different atoms of $f^{-i} \alpha$. Hence, for all $x \in F_{4}$,

$$
\#\left\{1+m \leq i \leq n+m: x \in B\left(\left(\partial f^{-i} \alpha, \delta / C\right)\right)\right\}>n \delta^{1 / 2}
$$

That is, points in $F_{4}$ are in at least $n \delta^{1 / 2}$ of the sets $\left\{f^{-i} B(\partial \alpha, \delta / C)\right\}_{i=1+m}^{n+m}$, a sequence of $n$ sets each with measure at most $C \delta / C=\delta$ by our condition on the measure of neighborhoods of the boundary of $\alpha$. Therefore, some $f^{-i^{*}} B(\partial \alpha, \delta / C)$ satisfies

$$
\delta>\mu\left(f^{-i^{*}} B(\partial \alpha, \delta / C)\right) \geq \mu\left(F_{4} \cap f^{-i^{*}} B(\partial \alpha, \delta / C)\right) \geq \frac{n \delta^{1 / 2}}{n} \mu\left(F_{4}\right)=\delta^{1 / 2} \mu\left(F_{4}\right)
$$

Hence, $\mu\left(F_{4}\right)<\delta^{1 / 2}$. Let

$$
\hat{F}_{4}=\bigcup_{A \in \omega:\left.\mu\right|_{A}\left(F_{4}\right)>\delta^{1 / 4}} A
$$

Again, by a similar argument to those above, $\mu\left(\hat{F}_{4}\right)<\delta^{1 / 4}$.
Assembling the full collection $\hat{F}=\hat{F}_{1} \cup \hat{F}_{2} \cup \hat{F}_{3} \cup \hat{F}_{4}$ of 'bad' atoms, for some constant $c$ we have $\mu(\hat{F})<c \delta^{1 / 4}=c \varepsilon$, as desired.

We now describe the construction of the sets $E_{1}$ and $E_{2}$ in Lemma 4.9. Consider an atom A of $\omega$ in the complement of $\hat{F}$. Recall, by the definition of $E_{1}$ that for $x \in E_{1}, h(x, \psi x)>\delta^{1 / 4}=\varepsilon$, so $x \in F_{4}$, and therefore $E_{1} \subset F_{4}$. Thus, since $A \in \hat{F}_{4}^{c},\left.\mu\right|_{A}\left(E_{1}\right)<\delta^{1 / 4}=\varepsilon$ as required.

To construct $E_{2}$, we will consider the union of various 'bad' subsets of the atom $A$. We first consider $A \cap R_{0}$; since $A \in \hat{F}_{1}^{c}$ and $\mu\left(R_{0}\right)<\delta$, we obtain $\left.\mu\right|_{A}\left(R_{0}\right)<c \delta$. We next consider the sets $A \cap F_{2}$ and $A \cap F_{3}$; as $A$ is in the complement of both $\hat{F}_{2}$ and $\hat{F}_{3}$ we readily obtain that $\left.\mu\right|_{A}\left(F_{2}\right)<\delta^{1 / 2}$ and $\left.\mu\right|_{A}\left(F_{3}\right)<\delta^{1 / 2}$.

The next sets consist of all the rectangles in which $A$ contains too large a proportion of either $F_{2}$ or $F_{3}$. We denote by $D_{2}$ the set that contains too large a proportion of $F_{2}$, measured either with respect to the product measure or the invariant measure.

Using the fact that $A$ is in the complement of $\hat{F}_{2}$, some computation, similar to the arguments given for bounding the measures $\mu\left(\hat{F}_{i}\right)$, shows that $\left.\mu\right|_{A}\left(D_{2}\right)<c \delta^{1 / 4}$. Similarly we define

$$
D_{3}=\bigcup_{R_{i}:\left.\mu\right|_{A \cap R_{i}}\left(F_{3}\right)>\delta^{1 / 4}} R_{i}
$$

and again, using the fact that $A$ is in the complement of $\hat{F}_{3}$, some computation shows that $\left.\mu\right|_{A}\left(D_{3}\right)<c \delta^{1 / 4}$.
The final bad subset is $\psi^{-1}\left(F_{2}\right)$. As we have already considered the intersection of $A$ with $R_{0}, F_{2}, F_{3}$, and $D_{2}$ we will only need to estimate the measure of the part of the subset that lives in the complement of these sets. Following the argument given in [CH96, §6.2], which estimates this measure for a particular $R_{i}$ in the complement of $R_{0} \cup D_{2}$ and the sums over all rectangles, we find that

$$
\left.\mu\right|_{A}\left(\psi^{-1}\left(F_{2}\right) \cap\left(R_{0} \cup F_{2} \cup F_{3} \cup D_{2}\right)^{c}\right) \leq c \delta
$$

We then define

$$
E_{2}=A \cap\left(R_{0} \cup F_{2} \cup F_{3} \cup D_{2} \cup D_{3} \cup \psi^{-1} F_{2}\right)
$$

Combining all the above estimates we see $\left.\mu\right|_{A}\left(E_{2}\right)<c \delta^{1 / 4}=c \varepsilon$.
We need the following fact, proven in section 6 of CH 96 , to complete the proof that $g_{t}$ is very weak Bernoulli.

Proposition 4.23 (Section 6, CH96). If $A$ is an atom with $A \subset \hat{F}_{2}{ }^{c} \cap \hat{F}_{3}{ }^{c}$ and $B \subset\left(A \cap R_{i} \cap E_{2}\right)^{c}$ for some rectangle $R_{i}$, then

$$
\left|\frac{\left.\mu\right|_{A \cap R_{i}}(B)}{\left.\mu\right|_{R_{i}}(\psi B)}-1\right|<c \delta^{1 / 4} .
$$

We now show how to use the above proposition to satisfy the conditions of Lemma 4.9. Consider a set $B \subset A \backslash E_{2}$. We will use Proposition 4.23 to bound $\left|\frac{\left.\mu\right|_{A}(B)}{\mu(\psi B)}-1\right|$.

First, we overestimate the term by breaking the sum up over rectangles contained in $\left(R_{0} \cup D_{2} \cup D_{3}\right)^{c}$ :

$$
\begin{aligned}
\left|\frac{\left.\mu\right|_{A}(B)}{\mu(\psi B)}-1\right| & \left.=\frac{1}{\mu(\psi B)}|\mu|_{A}(B)-\mu(\psi B) \right\rvert\, \\
& \leq\left.\sum_{R_{i} \subset\left(R_{0} \cup D_{2} \cup D_{3}\right)^{c}} \mu\right|_{\psi B}\left(R_{i}\right)|\underbrace{\frac{\left.\mu\right|_{A \cap R_{i}}(B)}{\left.\mu\right|_{R_{i}}(\psi B)}}_{1 s t} \cdot \underbrace{\frac{\left.\mu\right|_{A}\left(R_{i}\right)}{\mu\left(R_{i}\right)}}_{2 n d}-1|
\end{aligned}
$$

We know that for the second term,

$$
\left|\frac{\left.\mu\right|_{A}\left(R_{i}\right)}{\mu\left(R_{i}\right)}-1\right|<\delta
$$

since $\hat{F}_{1}$ is defined as the set where (3) fails and $A$ is in the complement of $\hat{F}_{1}$. As for the first term, directly from Proposition 4.23, we know that

$$
\left|\frac{\left.\mu\right|_{A \cap R_{i}}(B)}{\left.\mu\right|_{R_{i}}(\psi B)}-1\right|<c \delta^{1 / 4} .
$$

Combining these,

$$
1-(2 c+1) \delta^{1 / 4}<\left(1-c \delta^{1 / 4}\right)(1-\delta)<\frac{\left.\mu\right|_{A \cap R_{i}}(B)}{\left.\mu\right|_{R_{i}}(\psi B)} \cdot \frac{\left.\mu\right|_{A}\left(R_{i}\right)}{\mu\left(R_{i}\right)}<\left(1+c \delta^{1 / 4}\right)(1+\delta)<1-(2 c+1) \delta^{1 / 4}
$$

since $\delta$ is small. Finally,

$$
\begin{aligned}
\left|\frac{\left.\mu\right|_{A}(B)}{\mu(\psi B)}-1\right| & \leq\left.\sum_{R_{i} \subset\left(R_{0} \cup D_{2} \cup D_{3}\right)^{c}} \mu\right|_{\psi B}\left(R_{i}\right)\left|\frac{\left.\mu\right|_{A \cap R_{i}}(B)}{\left.\mu\right|_{R_{i}}(\psi(B))} \cdot \frac{\left.\mu\right|_{A}\left(R_{i}\right)}{\mu\left(R_{i}\right)}-1\right| \\
& <\left.(2 c+1) \delta^{1 / 4} \sum_{R_{i} \subset\left(R_{0} \cup D_{2} \cup D_{3}\right)^{c}} \mu\right|_{\psi B}\left(R_{i}\right) \\
& \leq(2 c+1) \delta^{\frac{1}{4}} .
\end{aligned}
$$

Recalling that $\varepsilon=\delta^{\frac{1}{4}}$, we have completed our verification of condition (2) of Lemma 4.9. Having satisfied the conditions of Lemma 4.9, we now have that

$$
\bar{d}\left(\left\{f^{-i} \alpha\right\}_{i=1+m}^{n+m},\left\{\left.f^{-i} \alpha\right|_{A}\right\}_{i=1+m}^{n+m}\right)<c^{\prime} \varepsilon,
$$

for some $c^{\prime}$ depending only on $G S$, as desired.

## References

[Ala22] Nawaf Alansari. Ergodic properties of measures with local product structure, 2022.
[BAPP19] Anne Broise-Alamichel, Jouni Parkkonen, and Frédéric Paulin. Equidistribution and counting under equilibrium states in negative curvature and trees, volume 329 of Progress in Mathematics. Birkhäuser/Springer, Cham, 2019.
[BCFT18] Keith Burns, Vaughn Climenhaga, Todd Fisher, and Dan Thompson. Unique equilibrium states for geodesic flows in nonpositive curvature. Geom. Funct. Anal., 28(5):1209-1259, 2018.
[BG89] Keith Burns and Marlies Gerber. Real analytic Bernoulli geodesic flows on $S^{2}$. Ergodic Theory Dynam. Systems, 9(1):27-45, 1989.
[BH99] Martin R. Bridson and André Haefliger. Metric spaces of non-positive curvature, volume 319 of Grundlehren der mathematischen Wissenschaften. Springer, Berlin, 1999.
$\left[\mathrm{CCE}^{+} 23\right]$ Benjamin Call, David Constantine, Alena Erchenko, Noelle Sawyer, and Grace Work. Unique equilibrium states for geodesic flows on flat surfaces with singularities. Int. Math. Res. Not. IMRN, (17):15155-15206, 2023.
[CH96] N. I. Chernov and C. Haskell. Nonuniformly hyperbolic K-systems are Bernoulli. Ergodic Theory Dynam. Systems, 16(1):19-44, 1996.
[Cli23] Vaughn Climenhaga. Gibbs measures have local product structure, 2023.
[CLT20a] David Constantine, Jean-François Lafont, and Dan Thompson. The weak specification property for geodesic flows on CAT(-1) spaces. Groups, Geometry, and Dynamics, 14:297-336, 2020.
[CLT20b] David Constantine, Jean-François Lafont, and Daniel J Thompson. Strong symbolic dynamics for geodesic flows on cat ( -1 ) spaces and other metric anosov flows. Journal de l'École polytechnique—Mathématiques, 7:201-231, 2020.
[CPZ20] Vaughn Climenhaga, Yakov Pesin, and Agnieszka Zelerowicz. Equilibrium measures for some partially hyperbolic systems. J. Mod. Dyn., 16, 2020.
[CT16] Vaughn Climenhaga and Daniel J. Thompson. Unique equilibrium states for flows and homeomorphisms with nonuniform structure. Adv. Math., 303:745-799, 2016.
[CT22] Benjamin Call and Daniel J. Thompson. Equilibrium states for self-products of flows and the mixing properties of rank 1 geodesic flows. J. Lond. Math. Soc. (2), 105(2):794-824, 2022.
[DT23] Caleb Dilsavor and Daniel J. Thompson. Gibbs measures for geodesic flow on cat(-1) spaces, 2023.
[LLS16] François Ledrappier, Yuri Lima, and Omri Sarig. Ergodic properties of equilibrium measures for smooth three dimensional flows. Comment. Math. Helv., 91(1):65-106, 2016.
[Orn70] Donald Ornstein. Imbedding Bernoulli shifts in flows. In Contributions to Ergodic Theory and Probability (Proc. Conf., Ohio State Univ., Columbus, Ohio, 1970), volume Vol. 160 of Lecture Notes in Math., pages 178-218. Springer, Berlin-New York, 1970.
[OW73] Donald Ornstein and Benjamin Weiss. Geodesic flows are Bernoullian. Israel J. Math, 14:184-198, 1973.
[OW98] Donald Ornstein and Benjamin Weiss. On the Bernoulli nature of systems with some hyperbolic structure. Ergodic Theory Dynam. Systems, 18(2):441-456, 1998.
[Pes77a] Ja. B. Pesin. Characteristic Ljapunov exponents, and smooth ergodic theory. Uspehi Mat. Nauk, 32(4(196)):55-112, 287, 1977.
[Pes77b] Ja. B. Pesin. Geodesic flows in closed Riemannian manifolds without focal points. Izv. Akad. Nauk SSSR Ser. Mat., 41(6):1252-1288, 1447, 1977.
[PVa19] Gabriel Ponce and Régis Varão. An introduction to the Kolmogorov-Bernoulli equivalence. SpringerBriefs in Mathematics. Springer, Cham, 2019. SBMAC SpringerBriefs.
[Rat74] M. Ratner. Anosov flows with Gibbs measures are also Bernoullian. Israel J. Math., 17:380-391, 1974.
[Rob03] Thomas Roblin. Ergodicité et équidistribution en courbure négative. Mém. Soc. Math. Fr. (N.S.), (95):vi+96, 2003.
Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL
E-mail address: bcall@uic.edu
Mathematics and Computer Science Department, Wesleyan University, Middletown, CT
E-mail address: dconstantine@wesleyan.edu
Department of Mathematics, Dartmouth College, Hanover, NH
E-mail address: alena.erchenko@dartmouth.edu
Mathematics and Computer Science Department, Southwestern University, Georgetown, TX
E-mail address: sawyern@southwestern.edu
Department of Mathematics, University of Wisconsin-Madison, Madison, WI
E-mail address: work2@wisc.edu

