UNIQUE EQUILIBRIUM STATES FOR GEODESIC FLOWS ON FLAT SURFACES WITH SINGULARITIES

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ABSTRACT. Consider a compact surface of genus ≥ 2 equipped with a metric that is flat everywhere except at finitely many cone points with angles greater than 2π. Following the technique in the work of Burns, Climenhaga, Fisher, and Thompson, we prove that sufficiently regular potential functions have unique equilibrium states if the singular set does not support the full pressure. Moreover, we show that the pressure gap holds for any potential which is locally constant on a neighborhood of the singular set. Finally, we establish that the corresponding equilibrium states have the K-property, and closed regular geodesics equidistribute.

1. INTRODUCTION

We examine the uniqueness of equilibrium states for geodesic flows on a specific class of CAT(0) surfaces, those where the negative curvature is concentrated at a finite set of points. Translation surfaces are examples of such surfaces. A translation surface X is a pair (X, ω) where X is a Riemann surface of genus g, and ω is a holomorphic one-form on X. The zeroes of this holomorphic 1-form occur at a finite set of points. The 1-form ω defines a metric which is flat everywhere except at its zeroes. At the zeros the metric has a conical singularity with angle $2(n+1)\pi$, where n is the order of the zero. For a more in-depth overview of translation surfaces see [Wri15, Zor06].

In [BCFT18], the authors prove that under certain conditions, a unique equilibrium state exists for potentials associated to the geodesic flow on a closed, rank 1 manifold with non-positive sectional curvature (an example of a CAT(0) space *without* singularities). The conditions are a Hölder continuous potential and a pressure gap, that is, topological pressure of the flow restricted to the singular set is strictly less than pressure of the flow overall. The singular set they consider is all the vectors in the unit tangent bundle with rank larger than one.

When the singular set is empty – for example in strictly negative curvature – every Hölder potential has a unique equilibrium state. When the singular set is non-empty, an additional condition is necessary as the geodesic flow is nonuniformly hyperbolic. Restricting the pressure of the flow on the singular set is a way of describing the flow of the singular set as having a small enough impact on the system as a whole that uniqueness is still guaranteed.

The natural way to define a geodesic flow on CAT(0) surfaces is to look at the flow on the set of all geodesics (see Section 2.1). Denote by GS the set of all geodesics on the surface S (see (2.1)). Then, a function $\phi\colon GS \to \mathbb{R}$ is called a *potential function* and an invariant Borel probability measure μ that maximizes the quantity $h_\mu(g_t) + \int_{GS} \phi \, d\mu$ (if it exists) is an *equilibrium state* for ϕ, where $h_\mu(g_t)$ is the measure-theoretic entropy with respect to the geodesic flow. The *pressure* for ϕ is the supremum of the quantity above when μ varies among invariant Borel probability measures for g_t.

In this paper, we study the uniqueness of equilibrium states for the geodesic flow described above, as we are guaranteed existence for continuous potentials by entropy expansivity of the flow [Ric19, Lemma 20]. In particular, we use the technique of [BCFT18] in our setting and define the singular set to be the set of geodesics which never encounter any cone points or, when they do, turn by angle exactly $\pm\pi$.

Remark. *Some other settings where the uniqueness of equilibrium states was studied are described in more detail below in the outline of the argument.*

We denote the singular set by Sing and the topological pressure of the potential $\phi|_{\text{Sing}}$ by $P(\text{Sing}, \phi)$, and prove the following:

Theorem A. *Let g_t be the geodesic flow on S, a compact surface of genus ≥ 2 equipped with a metric that is flat everywhere except at finitely many cone points which have angle greater than 2π. Let $\phi\colon GS \to \mathbb{R}$ be Hölder continuous. If $P(\text{Sing}, \phi) < P(\phi)$, then ϕ has a unique equilibrium state μ that has the K-property (see Definition 2.1).*

It is natural to ask for which potentials we have the *pressure gap* (i.e., the condition $P(\text{Sing}, \phi) < P(\phi)$) in Theorem A. The following theorem establishes the pressure gap for a sufficiently large class of Hölder continuous potentials, and thus uniqueness of equilibrium states.

Theorem B (Theorem 7.1 and Corollary 7.6)**.** *Let S, GS, and g_t be as in Theorem A. Let $\phi\colon GS \to \mathbb{R}$ be a Hölder continuous function which is locally constant on a neighborhood of* Sing*, or which is nearly constant. Then $P(\text{Sing}, \phi) < P(\phi)$.*

We slightly improve the case $\phi = 0$ from Ricks's result [Ric19, Theorem B] by showing that the unique measure of maximal entropy for the geodesic flow on S has the K-property which is stronger than mixing. Using the Patterson-Sullivan construction, Ricks builds a measure of maximal entropy μ [Ric17] and shows it is unique by asymptotic geometry arguments [Ric19]. We notice that Ricks's result holds for any compact, geodesically complete, locally CAT(0) space such that the universal cover admits a rank one axis.

We call a geodesic that is not in Sing *regular*. Using strong specification for a certain collection of 'good' orbit segments, we show that weighted regular closed geodesics equidistribute to these equilibrium states (see §8 for details).

Theorem C (Theorem 8.1)**.** *Let ϕ be as in Theorem B and μ_ϕ is the corresponding equilibrium state. Then, μ_ϕ is the weak* limit of weighted regular closed geodesics.*

1.1. **Outline of the argument.** A general scheme for proving that unique equilibrium states exist was developed by Climenhaga and Thompson in [CT16], building on ideas of Bowen in [Bow75] which were extended to flows in [Fra77]. To prove that there are unique equilibrium states for a flow $\{f_t\}$ and a potential ϕ on a compact metric space X, Climenhaga and Thompson ask for the following (see [CT16, Theorems A & C]):

- The pressure of obstructions to expansivity, $P^{\perp}_{exp}(\phi)$, is smaller than $P(\phi)$, and
- There are three collections of orbit segments $\mathcal{P}, \mathcal{G}, \mathcal{S}$, that we call collections of prefixes, good orbit segments, and suffixes, respectively, such that each orbit segment can be decomposed into a prefix, a good part, and a suffix (see [BCFT18, Definition 2.3]), satisfying

(I) $\mathcal{G}$ has the weak specification property at any scale,
(II) ϕ has the Bowen property on $\mathcal{G}$, and
(III) $P([\mathcal{P}] \cup [\mathcal{S}], \phi) < P(\phi)$.

This scheme was implemented for the geodesic flow on a closed rank 1 manifold with nonpositive sectional curvature in [BCFT18] and, more generally, without focal points in [CKP20, CKP19]. Also, it was used to obtain the uniqueness of the measure of maximal entropy on certain manifolds without conjugate points in [CKW19] and on CAT(-1) spaces in [CLT20b].

Our proof follows a specific approach to satisfying the conditions in the above scheme which was applied in [BCFT18], and which allows us to reduce condition (III) to checking the pressure of an invariant subset of GS. Although the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ is in general very abstract, we choose the decomposition using a function λ on the space of geodesics. We define this function, prove that it is lower semicontinuous, and describe how it gives rise to a decomposition in Section 3. For such a 'λ-decomposition', $\mathcal{P} = \mathcal{S}$ and, roughly speaking, orbit segments in $\mathcal{P}$ and $\mathcal{S}$ have small average values of λ wheareas any initial or terminal segment of an element of $\mathcal{G}$ has average value of λ which is not small. Furthermore, by utilizing a λ-decomposition, we are able to appeal to the following result:

Theorem 1.1 ([Cal20], Theorem 4.6)**.** *Let $\mathcal{F}$ be a continuous flow on a compact metric space X, and let $\phi : X \to \mathbb{R}$ be continuous. Suppose that flow is asymptotically entropy expansive, that $P^{\perp}_{\exp}(\phi) < P(\phi)$, and that $\lambda : X \to [0, \infty)$ is lower semicontinuous and bounded. If the λ-decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ satisfies the following:*

- *$\mathcal{G}(\eta)$ has strong specification at all scales, for all $\eta > 0$,*
- *ϕ has the Bowen property on $\mathcal{G}(\eta)$,*
- *$P(\bigcap_{t\in\mathbb{R}}(f_t \times f_t)\tilde{\lambda}^{-1}(0), \Phi) < 2P(\phi)$,*

where $\Phi(x, y) = \phi(x) + \phi(y)$ and $\tilde{\lambda}(x, y) = \lambda(x)\lambda(y)$, then $(X, \mathcal{F}, \phi)$ has a unique equilibrium state which has the K-property.

Theorem A will follow from Theorem 1.1 after we show that we can satisfy all conditions required. See Section 1.2 for the sections where each property is checked.

Our choice of λ gives a connection between orbit segments in $\mathcal{P}$ and $\mathcal{S}$ and the singular set Sing (see Definition 2.3). The singular set is also the source of the obstructions to expansivity (see Lemma 2.13). These connections are useful for proving the two 'pressure gap' properties Theorem 1.1 calls for: $P^{\perp}_{exp}(\phi) < P(\phi)$ and $P(\bigcap_{t\in\mathbb{R}}(f_t \times f_t)\tilde{\lambda}^{-1}(0), \Phi) < 2P(\phi)$. In particular, in our case $\bigcap_{t\in\mathbb{R}} f_t\lambda^{-1}(0) = \mathrm{Sing}$.

Remark. *The strong specification property on $\mathcal{G}$ in Theorem 1.1 is used to obtain that the equilibrium state has the K-property. The weak specification property on $\mathcal{G}$ is enough to guarantee the existence of a unique equilibrium state.*

Remark. *The K-property implies strong mixing of all orders.*

Remark. *We know that the geodesic flow is asymptotically entropy expansive in our setting, because the geodesic flow in CAT(0) spaces is entropy-expansive by* [Ric19, Lemma 20].

1.2. **Organization of the paper.** The paper is organized as follows. In Section 2 we provide definitions of and background on the main objects and tools of this paper and we record some basic geometric results which will be used throughout the paper. The main steps for the proof of Theorem A according to Theorem 1.1 are in Sections 3 (the λ-decomposition), 4 and 5 (the specification property for $\mathcal{G}$), and 6 (the Bowen property for $\mathcal{G}$).

We obtain Theorem B in Section 7, first proving the pressure gap condition for potentials which are locally constant on a neighborhood of Sing, and then using this result to note that the same gap holds for potentials with sufficiently small total variation. Theorem C (the equidistribution result) is proved in Section 8.

2. Background

2.1. **Setting and Definitions.** Throughout, S denotes a compact surface of genus ≥ 2 equipped with a metric which is flat everywhere except at finitely many conical points which have angles larger than 2π. Con denotes the set of conical points on S and denote by $\mathcal{L}(p)$ the total angle at a point $p \in S$. In particular, $\mathcal{L}(p) = 2\pi$ if $p \notin Con$ and $\mathcal{L}(p) > 2\pi$ if $p \in Con$. Denote by $\tilde{S}$ the universal cover of S, and note that $\tilde{S}$ is a complete CAT(0) space (see, e.g. [BH99] for definitions and basic results on CAT(0) spaces). Throughout, tildes denote the obvious lifts to the universal cover.

Let GS be the set of all (parametrized) geodesics in S. That is,

$$GS = \{\gamma\colon \mathbb{R} \to S : \gamma \text{ is a local isometry}\}. \tag{2.1}$$

We endow GS with the following metric:

$$d_{GS}(\gamma_1, \gamma_2) = \inf_{\tilde{\gamma}_1, \tilde{\gamma}_2} \int_{-\infty}^{\infty} d_{\tilde{S}}(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t)) e^{-2|t|}\, dt, \tag{2.2}$$

where the infimum is taken over all lifts $\tilde{\gamma}_i$ of γ_i to $G\tilde{S}$ for $i = 1, 2$. GS serves as an analogue of the unit tangent bundle in our setting. It is necessary to examine this more complicated space as geodesics in S are not determined by a tangent vector – they may branch apart from each other at points in Con. In this setting, the metric d_{GS} records the idea that two geodesics in GS are close if their images in S are nearby for all t in some large interval $[-T, T]$.

Geodesic flow on GS comes from shifting the parametrization of a geodesic:

$$(g_t\gamma)(s) = \gamma(s + t).$$

The normalizing factor 2 in our definition of d_{GS} serves to ensure that g_t is a unit-speed flow with respect to d_{GS}.

First, we recall a definition of the K-property of an invariant measure. See Section 10.8 in [CFS82] for a proof of the equivalence of this definition (known as K-mixing) with the original definition of the K-property, as well as more details about other equivalent definitions, such as completely positive entropy.

Definition 2.1. *A flow-invariant measure μ has the K-property if for all $t \neq 0$ for all $k \geq 1$ and all measurable sets $A_0, A_1, \ldots, A_k$ we have*

$$\lim_{n\to\infty} \sup_{B \in \mathcal{C}_n(A_1,\ldots,A_k)} |\mu(A_0 \cap B) - \mu(A_0)\mu(B)| = 0,$$

where $\mathcal{C}_n(A_1, \ldots, A_k)$ is the minimal σ-algebra generated by $g_{tr}(A_j)$ for $1 \leq j \leq k$ and natural $r \geq n$.

Remark. *The K-property implies strong mixing of all orders. We recall that an invariant measure μ is strongly mixing of all orders if for all $k \geq 1$ and all measurable sets $A_0, A_1, \dots, A_k$ we have*

$$\lim_{t_1\to\infty,\, t_{j+1}-t_j\to\infty} \mu(A_0 \cap g_{t_1}(A_1) \cap \ldots \cap g_{t_k}(A_k)) = \prod_{j=0}^{k} \mu(A_j).$$

A key tool in our analysis of the geodesic flow on S will be the turning angle of a geodesic at a cone point. We note that although S is not smooth at $p \in Con$, there is a well-defined space of directions at p, S_pS, and a well-defined notion of angle (see, e.g. [BH99, Ch. II.3]). In the angular metric, S_pS is a circle of total circumference $\mathcal{L}(p)$.

Definition 2.2. *Let $\gamma \in GS$. The turning angle of γ at time t is $\theta(\gamma, t) \in (-\frac{1}{2}\mathcal{L}(\gamma(t)), \frac{1}{2}\mathcal{L}(\gamma(t))]$ and is the signed angle between the segments $[\gamma(t-\delta), \gamma(t)]$ and $[\gamma(t), \gamma(t+\delta)]$ (for sufficiently small $\delta > 0$). A positive (resp. negative) sign for θ corresponds to a counterclockwise (resp. clockwise) rotation with respect to the orientation of $[\gamma(t-\delta), \gamma(t)]$.*

Since γ is a geodesic, $|\theta(\gamma, t)| - \pi \geq 0$ for any $t \in \mathbb{R}$. If $\gamma(t) \notin Con$, then $\theta(\gamma, t) = \pi$.

Definition 2.3. *We define the singular geodesics in S as*

$$\text{Sing} = \{\gamma \in GS : |\theta(\tilde{\gamma}, t)| = \pi \quad \forall t \in \mathbb{R}\}.$$

These are geodesics which either never encounter any cone points or, when they do, turn by angle exactly $\pm\pi$. These geodesics serve as an analogue of the singular set in the Riemannian setting of [BCFT18], i.e., geodesics which remain entirely in zero-curvature regions of the surface. In both cases the idea is that a singular geodesic never takes advantage of the geometric features of the surface (either its negative curvature regions or its large-angle cone points) to produce hyperbolic dynamical behavior. We note here a potentially confusing aspect of this terminology: a singular geodesic in this paper avoids the 'singular,' i.e. non-smooth, points of Con, or treats them as if they are not singular.

Below, we discuss some of the necessary definitions to apply the Climenhaga-Thompson machinery.

Definition 2.4. *Let $\varepsilon > 0$. The non-expansive set at scale ϵ for the flow g_t is*

$$\text{NE}(\varepsilon) = \{\gamma \in GS : \Gamma_\varepsilon(\gamma) \not\subset g_{[-s,s]}\gamma \text{ for all } s > 0\},$$

where

$$\Gamma_\varepsilon(\gamma) = \{\xi \in GS : d_{GS}(g_t\gamma, g_t\xi) \leq \varepsilon \quad \forall t \in \mathbb{R}\}.$$

The pressure of obstruction to expansivity for a potential ϕ is

$$P^\perp_{\exp}(\phi) = \lim_{\varepsilon\downarrow 0} \sup\{h_\mu(g_1) + \int_{GS} \phi\, d\mu \mid \mu(\text{NE}(\varepsilon)) = 1\},$$

where the supremum is taken over all g_t-invariant ergodic probability measures μ on GS such that $\mu(\text{NE}(\varepsilon)) = 1$.

In other words, a geodesic is in the complement of $\text{NE}(\epsilon)$ if the only geodesics which stay ϵ close to it for all time are contained in its own orbit. A flow is expansive if $\text{NE}(\epsilon)$ is empty for all sufficiently small ϵ. The presence of flat strips in our setting means our flow will not be expansive, but for small ϵ, the complement of $\text{NE}(\epsilon)$ will turn out to be a sufficiently rich set to use in our arguments.

In the interest of concision, we omit the formal definition of an orbit decomposition, referring instead to [CT16]. The key idea is the identification of a pair $(\gamma, t) \in GS \times [0, \infty)$ with the *orbit segment* $\{g_s\gamma \mid s \in [0, t]\}$. An orbit decomposition is a method of decomposing any orbit segment into three subsegments, a prefix, a central good segment, and a suffix. We denote the collections of these segments by $\mathcal{P}, \mathcal{G}$, and $\mathcal{S}$ respectively. The λ-decompositions that we use in this paper are orbit decompositions which decompose orbit segments based on a lower semicontinuous function λ.

We can define both specification and the Bowen property for an arbitrary collection of orbit segments $\mathcal{G} \subset GS \times [0, \infty)$. In both cases, by taking $\mathcal{G} = GS \times [0, \infty)$, one can retrieve the definitions for the full dynamical system.

Definition 2.5. *We say that $\mathcal{G}$ has weak specification if for all $\varepsilon > 0$, there exists $\tau > 0$ such that for any finite collection $\{(x_i, t_i)\}_{i=1}^n \subset \mathcal{G}$, there exists $y \in GS$ that ε-shadows the collection with transition times $\{\tau_i\}_{i=1}^n$ at most τ between orbit segments. In other words, for $1 \leq i \leq n$, there exists $\tau_i \in [0, \tau]$ and $y \in X$ such that*

$$d_{GS}(g_{t+s_i}y, g_t x_i) \leq \varepsilon \text{ for } 0 \leq t \leq t_i$$

where $s_k = \sum_{j=1}^{k-1} t_j + \tau_j$.

We say that $\mathcal{G}$ has strong specification when we can always take each $\tau_j = \tau$ in the above definition.

Definition 2.6. *Given a potential $\phi : GS \to \mathbb{R}$, we say that ϕ has the Bowen property on $\mathcal{G}$ if there is some $\varepsilon > 0$ for which there exists a constant $K > 0$ such that*

$$\sup \left\{ \left| \int_0^t \phi(g_r x) - \phi(g_r y)\, dr \right| \mid (x, t) \in \mathcal{G} \text{ and } d_{GS}(g_r y, g_r x) \leq \varepsilon \text{ for } 0 \leq r \leq t \right\} \leq K.$$

Remark. *If ϕ has the Bowen property on a collection of orbit segments $\mathcal{G}$ at some scale $\varepsilon > 0$, it in turn has the Bowen property on $\mathcal{G}$ at all smaller scales $\varepsilon' < \varepsilon$.*

There is also a definition of topological pressure for collections of orbit segments. However, by using Theorem 1.1, we sidestep this complication.

Finally, we adapt a piece of terminology from flat surfaces to our somewhat more general setting.

Definition 2.7. *A geodesic segment with both endpoints in Con and no cone points in its interior is called a saddle connection. A saddle connection path is composed of saddle connections joined so that the turning angle at each cone point is at least π. Note that with this definition all saddle connection paths are geodesics.*

2.2. **Basic geometric results.** In this section we collect a few basic results on the geometry of $S, \tilde{S}, GS$ and $G\tilde{S}$ which will be used in our subsequent arguments.

The following two lemmas relate the metric d_{GS} to the metric d_S on the surface itself, and will be useful for a number of our calculations below. First, we note that if two geodesics are close in GS, then they are close in S at time zero.

Lemma 2.8 ([CLT20b], Lemma 2.8)**.** *For all $\gamma_1, \gamma_2 \in GS$,*

$$d_S(\gamma_1(0), \gamma_2(0)) \leq 2 d_{GS}(\gamma_1, \gamma_2).$$

Furthermore, for $s, t \in \mathbb{R}$, $d_S(\gamma_1(s), \gamma_2(t)) \leq 2 d_{GS}(g_s\gamma_1, g_t\gamma_2)$.

Conversely, if two geodesics are close in S for a significant interval of time surrounding zero, then they are close in GS:

Lemma 2.9 ([CLT20b], Lemma 2.11)**.** *Let ϵ be given and $a < b$ arbitrary. Then, there exists $T = T(\epsilon) > 0$ such that if $d_S(\gamma_1(t), \gamma_2(t)) < \epsilon/2$ for all $t \in [a-T, b+T]$, then $d_{GS}(g_t\gamma_1, g_t\gamma_2) < \epsilon$ for all $t \in [a,b]$. For small ϵ, we can take $T(\epsilon) = -\log(\epsilon)$.*

A similar, and more specialized result which we will need later in the paper (see the proof of Proposition 6.2) is the following

Lemma 2.10. *Suppose that $d_S(\gamma_1(t), \gamma_2(t)) = 0$ for all $t \in [a,b]$. Then, for all $t \in [a,b]$, $d_{GS}(g_t\gamma_1, g_t\gamma_2) \leq e^{-2\min\{|t-a|,|t-b|\}}$.*

Proof. For any $x \geq 0$, $\int_x^\infty (s-x)e^{-2s}ds = \frac{1}{4}e^{-2x}$. In the setting of the Lemma, since the distance between the geodesics is zero on $[a,b]$ and since geodesics move at unit speed,

$$d_{GS}(g_t\gamma_1, g_t\gamma_2) \leq \int_{-\infty}^{a} 2(a-s)e^{-2|t-s|}ds + \int_b^\infty 2(s-b)e^{-2|t-s|}ds.$$

Quick changes of variables show that this is equal to $\int_{|t-a|}^\infty 2(s-|t-a|)e^{-2s}ds + \int_{|t-b|}^\infty 2(s-|t-b|)e^{-2s}ds = \frac{1}{2}(e^{-2|t-a|} + e^{-2|t-b|})$, and the Lemma follows. ☐

The geodesic flow has the following Lipschitz property:

Lemma 2.11 ([CLT20a], Lemma 2.5)**.** *Fix a $T > 0$. Then, for any $t \in [0,T]$, and any pair of geodesics $\gamma, \xi \in GS$,*

$$d_{GS}(g_t\gamma, g_t\xi) < e^{2T} d_{GS}(\gamma, \xi).$$

We need the following four geometric facts, which are basic consequences of the compactness of S:

Lemma 2.12. *Under the assumption $Con \neq \emptyset$,*

(a) *There exists some $d_0 > 0$ such that $\tilde{S}$ contains no flat $d_0 \times d_0$ square.*
(b) *There exists some $\eta_0 > 0$ such that the excess angle at every cone point in S is at least η_0.*
(c) *There exists some $\ell_0 > 0$ such that the length of every saddle connections is at least ℓ_0.*
(d) *There exists some $\theta_0 > 0$ such that the excess angle at every cone point in S is at most θ_0.*

We note here that Sing is the source of the non-expansivity for our geodesic flow:

Lemma 2.13. *For all sufficiently small $\varepsilon > 0$, $\mathrm{NE}(\varepsilon) \subset \mathrm{Sing}$.*

Proof. Suppose $\gamma \in \mathrm{NE}(\varepsilon)$ and that ϵ is smaller than half the injectivity radius of S. Then, there exists $\xi \in GS$ which is not in the orbit of γ such that $d_{GS}(g_t\gamma, g_t\xi) \leq \varepsilon$ for all $t \in \mathbb{R}$. By Lemma 2.8, $d_S(\gamma(t), \xi(t)) \leq 2\varepsilon$ for all $t \in \mathbb{R}$. In particular, using our assumption on ϵ, there exist lifts $\tilde{\gamma}$ and $\tilde{\xi}$ such that $d_{\tilde{S}}(\tilde{\gamma}(t), \tilde{\xi}(t)) \leq 2\varepsilon$ for all $t \in \mathbb{R}$. By the Flat Strip Theorem ([Bal95, Corollary 5.8 (ii)]), there is an isometric embedding $\mathbb{R} \times [a,b] \to \tilde{S}$ sending $\mathbb{R} \times \{a\}$ to the image of $\tilde{\gamma}$ and $\mathbb{R} \times \{b\}$ to the image of $\tilde{\xi}$.

Since $\tilde{\xi}$ is not in the orbit of $\tilde{\gamma}$, we must have $a \neq b$ and the isometrically embedded strip is non-degenerate. But this immediately implies that for all t, $|\theta(\tilde{\gamma}, t)| = \pi$ as $\tilde{\gamma}$ always turns at angle π on the side to which the embedded flat strip lies. Therefore, $\gamma \in \mathrm{Sing}$. ☐

Lemma 2.14. *Given any closed geodesic $\gamma \subset S$, there is a closed saddle connection path which is homotopic to γ and has the same length as γ.*

Proof. Assume γ contains a point $p \in Con$. Then, the desired closed saddle connection path is the geodesic that starts at p and traces γ.

Suppose $\gamma \subset S \setminus Con$, and so $\tilde{\gamma} \subset \tilde{S} \setminus \widetilde{Con}$. Fix an orientation of $\tilde{\gamma}$ and consider the variation $\tilde{\gamma}_t$ of curves given by sliding $\tilde{\gamma}$ to its left (so the variational field is perpendicular to $\tilde{\gamma}$ and to its left with respect to $\tilde{\gamma}$'s orientation). For as long as $\tilde{\gamma}_t$ lies in $\tilde{S} \setminus \widetilde{Con}$, it is still a geodesic, as $\tilde{S} \setminus \widetilde{Con}$ is flat. Furthermore, if γ_t is its image in S, the length of γ_t is the same as the length of γ, again using the flat geometry of $\tilde{S} \setminus \widetilde{Con}$.

If $\tilde{\gamma}_t$ is defined for all $t > 0$, then an entire half-space of $\tilde{S}$ is in $\tilde{S} \setminus \widetilde{Con}$. But this implies that S itself is flat, a contradiction. Then, there must be $t^* > 0$ so that as $t \to t^*$ from below, $\tilde{\gamma}_t$ limits on some geodesic containing at least one point in $\widetilde{Con}$ with the same length as $\tilde{\gamma}$, and the image of this curve in S (with appropriate parametrization) is the saddle connection path we want.

□

3. The λ-decomposition

We now turn to the main arguments of the paper. First, following the ideas in [BCFT18], we establish the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ as a 'λ-decomposition' using the function λ in Definition 3.3 which is defined through two auxiliary functions that view the stable and unstable parts of any given geodesic. Throughout this section, fix $s > 0$ such that $2s$ is less than the shortest saddle connection of S. Below we omit in the notation the dependence of functions on s.

Definition 3.1. *We define $\lambda^{uu} \colon GS \to [0, \infty)$ by*

$$\lambda^{uu}(\gamma) = \frac{|\theta(\gamma, c)| - \pi}{\max\{s, c\}},$$

where $c \geq 0$ is the first time that $\gamma(c)$ hits a cone point and turns with angle strictly greater than π (naturally, we set $\lambda^{uu}(\gamma) = 0$ in case $c = \infty$).

Definition 3.2. *We define $\lambda^{ss} \colon GS \to [0, \infty)$ by*

$$\lambda^{ss}(\gamma) = \frac{|\theta(\gamma, c)| - \pi}{\max\{s, |c|\}},$$

where $c \leq 0$ is the most recent time that $\gamma(c)$ has hit a cone point and turned with angle strictly greater than π (naturally, we set $\lambda^{ss}(\gamma) = 0$ in case $c = -\infty$).

We now define our function λ so that near cone points at which geodesics turn with angle greater than π, it measures the turning angle at that cone point (multiplied by a constant), and far from a cone point, it measures both distance and turning angle from both the previous and next cone point.

Definition 3.3. *Let λ^{uu} and λ^{ss} be functions defined in Definitions 3.1 and 3.2, respectively. We define $\lambda\colon GS \to [0,\infty)$ by*

$$\lambda(\gamma) = \begin{cases} \lambda^{ss}(\gamma) & \text{if there exists } c \in (-s,0] \text{ such that } |\theta(\gamma,c)| - \pi > 0, \\ \lambda^{uu}(\gamma) & \text{if there exists } c \in [0,s) \text{ such that } |\theta(\gamma,c)| - \pi > 0, \\ \min\{\lambda^{ss}(\gamma), \lambda^{uu}(\gamma)\} & \text{otherwise.} \end{cases}$$

Observe that it is well-defined when $\gamma(0)$ is a cone point, as in that case, $\lambda^{uu}(\gamma) = \lambda^{ss}(\gamma)$.

We prove several properties of the defined λ.

Proposition 3.4. *If $\lambda(\gamma) = 0$, then $\lambda(g_t\gamma) = 0$ either for all $t \geq 0$ or for all $t \leq 0$.*

Proof. If $\lambda(\gamma) = 0$, then γ does not turn at a cone point in the interval $(-s,s)$, and so, $\lambda^{uu}(\gamma) = 0$ or $\lambda^{ss}(\gamma) = 0$. In the first case, this implies that γ never turns at a cone point in the future. Therefore, for all $t \geq 0$, $\lambda(g_t\gamma) = \lambda^{uu}(g_t\gamma) = 0$. A similar argument holds with $t \leq 0$ if $\lambda^{ss}(\gamma) = 0$. □

As a corollary, we have

Corollary 3.5. $\bigcap_{t\in\mathbb{R}} g_t\lambda^{-1}(0) = \mathrm{Sing}$.

Furthermore, this allows us to show that the pressure gap for the product flow (condition (3) of Theorem 1.1) is implied by the pressure gap $P(\mathrm{Sing},\phi) < P(\phi)$ that we will establish in §7.

Proposition 3.6. *(Following* [CT19, Proposition 5.1]*) If $P(\mathrm{Sing},\phi) < P(\phi)$, then $P(\bigcap_{t\in\mathbb{R}}(g_t \times g_t)(\tilde{\lambda})^{-1}(0), \Phi) < 2P(\Phi)$, where $\Phi(x,y) = \phi(x) + \phi(y)$ and $\tilde{\lambda}(x,y) = \lambda(x)\lambda(y)$.*

Proof. Let ν be an invariant measure supported on $\bigcap_{t\in\mathbb{R}}(g_t \times g_t)(\tilde{\lambda})^{-1}(0)$, and let

$$A = \bigcap_{t\in\mathbb{R}}(g_t \times g_t)(\tilde{\lambda})^{-1}(0) \cap (\mathrm{Reg} \times \mathrm{Reg}).$$

We will show that $\nu(A) = 0$ by showing that it contains no recurrent points. Assume for contradiction that $(\gamma_1,\gamma_2) \in A$ is a recurrent point, and then assume without loss of generality that $\lambda(\gamma_1) = 0$. Since $\gamma_1 \notin \mathrm{Sing}$, it follows that $d(\gamma_1,\mathrm{Sing}) = c > 0$, which from recurrence, implies that there exists a sequence $t_k \to \infty$ such that $d(g_{t_k}\gamma_1,\mathrm{Sing}) > \frac{c}{2}$, with a similar claim holding in backwards time. However, we also know that $d(g_t\gamma_1,\mathrm{Sing}) \to 0$ as $t \to \infty$, or as $t \to -\infty$ by Proposition 3.4. Thus, we have arrived at a contradiction. Hence, ν is supported on the complement of $\mathrm{Reg} \times \mathrm{Reg}$, which is $(\mathrm{Sing} \times GS) \cup (GS \times \mathrm{Sing})$, and so, $P_\nu(\Phi) \leq P(\mathrm{Sing},\phi) + P(\phi)$. By the Variational Principle, our proof is complete. □

We have also constructed λ so that it is lower semi-continuous.

Lemma 3.7. *Let $s > 0$ be such that $2s$ is less than the shortest saddle connection of S. Then, λ defined in Definition 3.3 is lower semicontinuous.*

Proof. Let $\gamma \in GS$. We show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\lambda(\gamma) - \varepsilon < \lambda(\xi)$ for all $\xi \in GS$ such that $d_{GS}(\gamma,\xi) < \delta$. To ease the arguments below slightly, we work in $\tilde{S}$ with lifts $\tilde{\gamma},\tilde{\xi}$ so that $d_{G\tilde{S}}(\tilde{\gamma},\tilde{\xi}) = d_{GS}(\gamma,\xi)$. Recall that by Lemma 2.8, if $d_{G\tilde{S}}(\tilde{\gamma},\tilde{\xi}) < \delta$ then $d_{\tilde{S}}(\tilde{\gamma}(0),\tilde{\xi}(0)) < 2\delta$.

If $\lambda(\gamma) = 0$, then we are done as λ is a non-negative function. Therefore, for the rest of the argument we assume that $\lambda(\gamma) > 0$.

Case 1: Suppose there exists $c \in (-s, s)$ such that $\psi := |\theta(\gamma, c)| - \pi > 0$. Denote $\tilde{\gamma}(c) = p$. We show that there exists $\delta > 0$ such that $p \in \tilde{\xi}((-s, s))$.

Let C_1 be the cone with vertex p generated by the vectors that make angle $\leq \frac{\psi}{2}$ with the segment $\tilde{\gamma}((c, s))$ and let C_2 be the cone with the vertex p generated by the vectors that make angle $\leq \frac{\psi}{2}$ with the axis $\gamma((-s, c))$. (See Figure 1.) Then there exist u_1, u_2 and $\delta_1, \delta_2 > 0$ such that $B_1 := B(\tilde{\gamma}(u_1), \delta_1) \subset C_1 \cap B(\tilde{\gamma}(0), s)$ and $B_2 := B(\tilde{\gamma}(u_2), \delta_2) \subset C_2 \cap B(\tilde{\gamma}(0), s)$. By Lemmas 2.8 and 2.11, there exists $\delta > 0$ such that if $d_{GS}(\gamma, \xi) < \delta$ then $\tilde{\xi}$ passes through B_1 and B_2. Since any two points in a CAT(0)-space are connected by a unique geodesic segment and by our construction of C_1 and C_2, we obtain that if $d_{GS}(\gamma, \xi) < \delta$ then $\tilde{\xi}$ passes through p. Further shrinking δ as necessary so that $\delta + d_S(\gamma(0), p) < s$, by Lemma 2.8 and the triangle inequality for the triangle with vertices $\tilde{\xi}(0), \tilde{\gamma}(0)$, and p, we have $p \in \tilde{\xi}((-s, s))$ if $d_{GS}(\gamma, \xi) < \delta$. Let $t_0 \in (-s, s)$ be such that $\tilde{\xi}(t_0) = p$. Moreover, $|t_0 - c| \leq 2\delta$.

By the triangle inequality,

$$d_{\tilde{S}}(\tilde{\xi}(t_0+t), \tilde{\gamma}(c+t)) \leq d_{\tilde{S}}(\tilde{\xi}(t_0+t), \tilde{\xi}(c+t)) + d_{\tilde{S}}(\tilde{\xi}(c+t), \tilde{\gamma}(c+t)) = |t_0 - c| + d_{\tilde{S}}(\tilde{\xi}(c+t), \tilde{\gamma}(c+t)).$$

Let $\tilde{\xi}_1 = g_{t_0}\tilde{\xi}$ and $\tilde{\gamma}_1 = g_c\tilde{\gamma}$. Then, by the above inequality and Lemma 2.11,

$$d_{G\tilde{S}}(\tilde{\xi}_1, \tilde{\gamma}_1) \leq 2\delta + e^{2|c|}\delta = (2 + e^{2|c|})\delta. \tag{3.1}$$

Moreover, for all $t \in (0, s - c]$, we obtain that

$$d_{\tilde{S}}(\tilde{\xi}_1(t), \tilde{\gamma}_1(t)) = \begin{cases} 2t & \text{if} \quad \alpha \geq \pi, \\ 2t\sin(\alpha/2) & \text{if} \quad 0 \leq \alpha \leq \pi, \end{cases}$$

where α is the (unsigned) angle between the outward trajectories of $\tilde{\gamma}_1$ and $\tilde{\xi}_1$ from the cone point p.

If $\alpha \geq \pi$, then $d_{GS}(\xi_1, \gamma_1) \geq \int_0^{s-c} 2te^{-2t}dt$, which is not possible for sufficiently small δ by (3.1).

Consider $\alpha \in [0, \pi)$. Then, we have that

$$\sin(\alpha/2) < \delta(2 + e^{2|c|}) \left(\int_0^{s-c} 2te^{-2t}dt \right)^{-1}. \tag{3.2}$$

Let β be the (unsigned) angle between the inward trajectories $\tilde{\gamma}_1$ and $\tilde{\xi}_1$ at p. Similarly to the argument above, we obtain that for sufficiently small δ,

$$\sin(\beta/2) < \delta(2 + e^{2|c|}) \left(-\int_{-s-c}^{0} 2te^{2t}dt \right)^{-1}. \tag{3.3}$$

Using (3.2) and (3.3),

$$|\lambda(\gamma) - \lambda(\xi)| = \frac{1}{s}||\theta(\tilde{\gamma}, c)| - |\theta(\tilde{\xi}, t_0)|| \leq \frac{1}{s}(\alpha + \beta) \leq C\delta,$$

where C depends only on s and c. Thus, for sufficiently small δ, we have $|\lambda(\gamma) - \lambda(\xi)| < \varepsilon$.

Case 2: Assume there exists $c_1 \leq -s$ and $c_2 \geq s$ such that $\psi_1 := |\theta(\gamma, c_1)| - \pi > 0$ and $\psi_2 := |\theta(\gamma, c_2)| - \pi > 0$. Denote $\tilde{\gamma}(c_1) = p_1$ and $\tilde{\gamma}(c_2) = p_2$.

Let C_1 be the cone with vertex p_1 generated by the vectors that make angle $\leq \frac{\min\{\psi_1, \psi_2\}}{2}$ with the segment $\tilde{\gamma}((c_1, -s])$ if $c_1 \neq -s$ or $\tilde{\gamma}((-2s, -s))$ otherwise. Let C_2 be the cone with vertex p_2 generated by the vectors that make angle $\leq \frac{\min\{\psi_1, \psi_2\}}{2}$ with the segment $\tilde{\gamma}([s, c_2])$ if $c_2 \neq s$ or $\tilde{\gamma}((s, 2s))$ otherwise. Similar to Case 1, we have that, by Lemmas 2.8 and 2.11, there exists $\delta > 0$ such that if $d_{GS}(\gamma,\xi) < \delta$ then $\tilde{\xi}$ passes through p_1 and p_2. In particular, $\tilde{\gamma}$ and $\tilde{\xi}$ share a geodesic connecting p_1 and p_2. Therefore, there exists d such that $g_d\tilde{\xi}(t) = \tilde{\gamma}(t)$ for $t \in [c_1, c_2]$. Let t_1 and t_2 be such that $\tilde{\xi}(t_1) = p_1$ and $\tilde{\xi}(t_2) = p_2$. Then, $|t_1 - c_1| \le 2e^{2|c_1|}\delta$ and $|t_2 - c_2| \le 2e^{2c_2}\delta$ so $|d| \le 2e^{2\min\{|c_1|,c_2\}}\delta$. Moreover, by the triangle inequality,

$$d_{G\tilde{S}}(g_d\tilde{\xi},\gamma) \le (2e^{2\min\{|c_1|,c_2\}} + 1)\delta. \tag{3.4}$$

Let α_1 and α_2 be the (unsigned) angles between the inward and outward trajectories of $g_d\tilde{\xi}$ and $\tilde{\gamma}$ at p_1 and p_2, respectively. Similarly to Case 1, we have $0 \le \alpha_1, \alpha_2 \le \pi$,

$$\sin(\alpha_1/2) \le \delta(2e^{2\min\{|c_1|,c_2\}} + 1)\left(-\int_{c_2}^{\infty} 2te^{2t}dt\right)^{-1},$$

and

$$\sin(\alpha_2/2) \le \delta(2e^{2\min\{|c_1|,c_2\}} + 1)\left(\int_{c_1}^{\infty} 2te^{-2t}dt\right)^{-1}$$

Therefore,

$$|\lambda^{ss}(\gamma) - \lambda^{ss}(\xi)| \le C\delta \quad \text{and} \quad |\lambda^{uu}(\gamma) - \lambda^{uu}(\xi)| \le C\delta,$$

where C depends only on S, c_1, and c_2.

Thus, if $t_1 = c_1 + d \le -s$ and $t_2 = c_2 + d \ge s$, then $\lambda(\xi) = \min\{\lambda^{ss}(\xi), \lambda^{uu}(\xi)\}$ and we have $|\lambda(\gamma) - \lambda(\xi)| \le C\delta$.

Otherwise, for sufficiently small δ, $\lambda(\xi) \ge \min\{\lambda^{ss}(\xi), \lambda^{uu}(\xi)\}$ and we have $\lambda(\xi) \ge \lambda(\gamma) - C\delta$. □

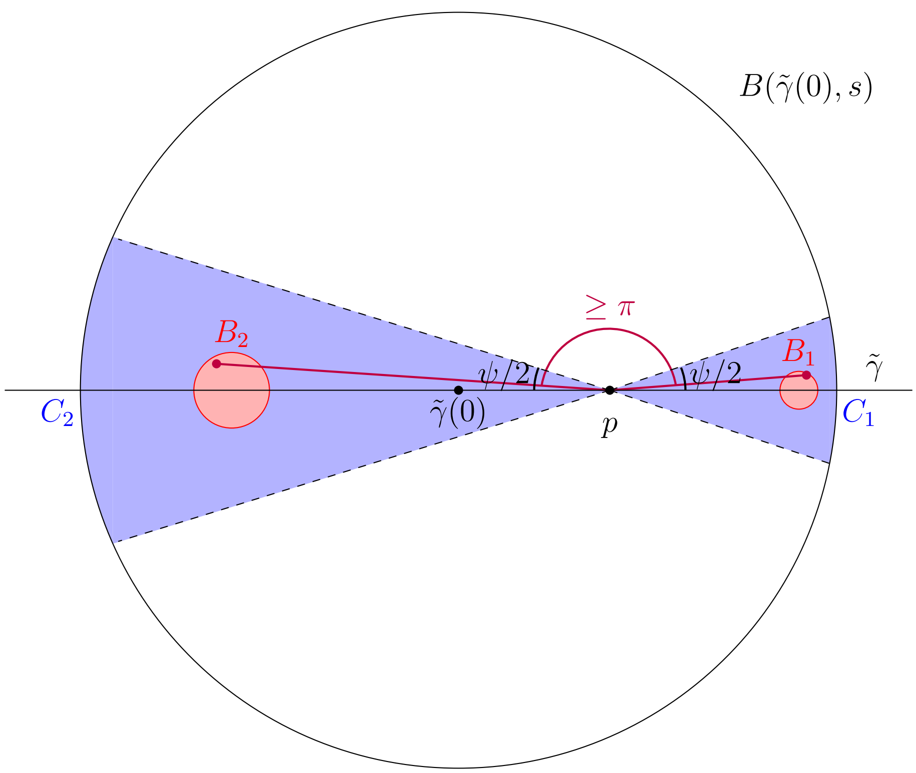

FIGURE 1. The argument for Case 1 in Lemma 3.7. The geodesic segments connecting points in B_2 and B_1 meet at the cone point p with angle $\ge \pi$ on both sides. Any geodesic connecting points in B_2 and B_1 must run through p.

Following §3 of [BCFT18], or Definition 3.4 in [CT19], we define

$$\mathcal{G}(\eta) = \left\{(\gamma,t) | \int_0^\rho \lambda(g_u(\gamma))du \geq \eta\rho \quad \text{and} \quad \int_0^\rho \lambda(g_{-u}g_t(\gamma))du \geq \eta\rho \quad \text{for} \quad \rho \in [0,t]\right\}$$

and

$$\mathcal{B}(\eta) = \left\{(\gamma,t) | \int_0^\rho \lambda(g_u(\gamma))du < \eta\rho\right\}.$$

The decomposition we will take is $(\mathcal{P},\mathcal{G},\mathcal{S}) = (\mathcal{B}(\eta),\mathcal{G}(\eta),\mathcal{B}(\eta))$ for a sufficiently small value of η which will be determined below.

While near cone points, positivity of λ only gives us information about the closest cone point, and far from cone points, it gives us information about cone points on both sides. The following propositions help us quantify these relationships. Let θ_0 be as in Lemma 2.12.

Proposition 3.8. *There exist functions $f, M : (0,\infty) \to (0,\infty)$ such that if $\lambda(\gamma) > \eta$, then there is a cone point in $\gamma[-M(\eta), M(\eta)]$ with turning angle at least $f(\eta)$ away from $\pm\pi$.*

Proof. Suppose $\lambda(\gamma) > \eta$. First, we find $M(\eta)$. The furthest cone point can be (in future or past) when γ turns with the maximum possible angle at that point, which is $\theta_0/2$. Thus, we see that

$$\eta \leq \lambda(\gamma) \leq \frac{|\theta(\gamma,c)| - \pi}{c} \leq \frac{\theta_0/2}{c}$$

and so $c \leq \frac{\theta_0}{2\eta}$. The smallest angle that γ can turn at a cone point is when the cone point is very close to 0. In this case, we see that

$$\eta \leq \frac{|\theta(\gamma,c)| - \pi}{s}$$

and so $s\eta \leq |\theta(\gamma,c)| - \pi$. Thus, we can take $M(\eta) = \frac{\theta_0}{2\eta}$ and we can take $f(\eta) = s\eta$. □

Corollary 3.9. *Let $\eta > 0$. If $(\gamma,t) \in \mathcal{G}(\eta)$, then there exists $t_1, t_2 \in [-\frac{\theta_0}{2\eta}, \frac{\theta_0}{2\eta}]$ such that $\gamma(t_1), \gamma(t+t_2) \in \mathrm{Con}$, and furthermore, the turning angles at these cone points are at least $s\eta$.*

4. $\mathcal{G}(\eta)$ HAS WEAK SPECIFICATION (AT ALL SCALES)

The goal of this section is to obtain Corollary 4.5 which shows that $\mathcal{G}(\eta)$ has weak specification at all scales.

Lemma 4.1. *(Compare with Lemma 3.8 in* [Dan11]*) Let $x \in S$ and β be an outgoing direction at x. Then, for any $\varepsilon > 0$ there exist $T_0(\varepsilon)$ and a geodesic c which connects x with a point $z \in Con$ so that the length of c is at most $T_0(\varepsilon)$ and $\angle_x(\beta, c) < \varepsilon$ where $\angle_x(a,b)$ is the angle at x between the directions a and b.*

Proof. Let $\tilde{x} \in \tilde{S}$ be a lift of x. Denote by $C_{\frac{\varepsilon}{2}}(\tilde{x},\beta)$ the $\frac{\varepsilon}{2}$-cone around the β direction centered at $\tilde{x}$ in $\tilde{S}$. There exists $T_0 = T_0(\varepsilon)$ such that $I = B_{T_0}(\tilde{x}) \cap C_{\frac{\varepsilon}{2}}(\tilde{x},\beta)$ contains a fundamental domain of S. Then $\widetilde{Con} \cap Int(I) \neq \emptyset$, so let $\tilde{z} \in \widetilde{Con} \cap Int(I)$ such that $\tilde{z}$ is a closest to $\tilde{x}$. The segment $\tilde{c} = \tilde{x}\tilde{z}$ is a geodesic of length at most T_0. The projection of $\tilde{c}$ to S is the desired geodesic. □

Lemma 4.2. *For any $\delta > 0$, there exists $T_1 = T_1(\delta, S)$ such that for any $t > 0$ and $(\gamma, t) \in \mathcal{G}(\eta)$ there exists a saddle connection path γ_e such that $\ell(\gamma_e) \leq t + 2T_1$ and there exists $s_0 \in [0, T_1]$ with the property that if γ_e^c is any extension of γ_e to a complete geodesic then $d_{GS}(g_u(\gamma), g_u(g_{s_0}(\gamma_e^c))) \leq \delta$ for all $u \in [0, t]$. In particular, if $t > \frac{\theta_0}{\eta}$, there exists a closed interval $I \supset [\frac{\theta_0}{2\eta}, t - \frac{\theta_0}{2\eta}]$ such that $\gamma_e(s_0 + u) = \gamma(u)$ for $u \in I$.*

Proof. As usual, we prove the result in $\tilde{S}$. Let $T = \max\{-\log(\delta), \frac{\theta_0}{2\eta}\}$. By Lemma 2.9 if we construct $\tilde{\gamma}_e$ such that $d_{\tilde{S}}(\tilde{\gamma}(u), \tilde{\gamma}_e(s_0+u)) < \frac{\delta}{2}$ for all $u \in [-T, t+T]$, then $d_{GS}(g_u(\gamma), g_u(g_{s_0}(\gamma_e^c))) \leq \delta$ for all $u \in [0, t]$.

By Corollary 3.9, there exist $t_0, t_1 \in [-\frac{\theta_0}{2\eta}, \frac{\theta_0}{2\eta}]$ such that $\tilde{\gamma}(t_0), \tilde{\gamma}(t + t_1) \in \widetilde{Con}$, $|\theta(\tilde{\gamma}, t_0)| - \pi \geq s\eta$ and $|\theta(\tilde{\gamma}, t + t_1)| - \pi \geq s\eta$. Thus, there exist $s_1 \in [-T, \frac{\theta_0}{2\eta}]$ and $s_2 \in [-\frac{\theta_0}{2\eta}, T]$ such that $\tilde{\gamma}(s_1), \tilde{\gamma}(t + s_2) \in \widetilde{Con}$ and $(\tilde{\gamma}([-T, s_1)) \cup \tilde{\gamma}((t + s_2, t + T])) \cap \widetilde{Con} = \emptyset$.

If $s_1 = -T$, then define $\tilde{\gamma}_e(u - s_1) = \tilde{\gamma}(u)$ for $u \in [s_1, t + s_2]$. Assume $s_1 > -T$. Let η_0 be as in Lemma 2.12. Choose $\alpha < \frac{\ell_0}{4(T + \frac{\theta_0}{2\eta})} \min\{\eta_0, \frac{\delta}{2(\frac{\theta_0}{2\eta} + T)}\}$, where ℓ_0 is as in Lemma 2.12. Let $\mathcal{C}$ be the cone in $\tilde{S}$ containing $\tilde{\gamma}([-T, s_1])$ with vertex angle α such that any geodesic segment in $\mathcal{C}$ can be concatenated with $\tilde{\gamma}([s_1, t])$ to form a geodesic. By Lemma 4.1, there exists $T_0 = T_0(\frac{\alpha}{2}) \geq T + \frac{\theta_0}{2\eta}$ and a point $\tilde{p}_1$ in $\widetilde{Con} \cap \mathcal{C}$ such that $d_{\tilde{S}}(\tilde{p}, \tilde{\gamma}(s_1)) \leq T_0$. Choose $\tilde{p}_1$ as in the previous sentence minimizing the distance to $\tilde{\gamma}([-T_0, s_1])$. If $d_{\tilde{S}}(\tilde{p}_1, \tilde{\gamma}(s_1)) \geq T + \frac{\theta_0}{2\eta}$, then let the initial segment of $\tilde{\gamma}_e$ be the geodesic segment $[\tilde{p}_1, \tilde{\gamma}(s_1)]$.

Otherwise, we repeat the argument above, applying Lemma 4.1 to construct an α-cone centered around the geodesic segment making angle $\pi + \frac{\alpha}{2}$ with $[\tilde{p}_1, \tilde{\gamma}(s_1)]$. We get a point $\tilde{p}_2 \in \widetilde{Con}$ in this cone with $d_{\tilde{S}}(\tilde{p}_1, \tilde{p}_2) \leq T_0$, again chosen to minimize the distance to $\tilde{\gamma}([-T_0, s])$. If $d_{\tilde{S}}(\tilde{p}_2, \tilde{\gamma}(s_1)) \geq T + \frac{\theta_0}{2\eta}$, then let the initial segment of $\tilde{\gamma}_e$ be the concatenation of geodesic segments $[\tilde{p}_2, \tilde{p}_1]$ and $[\tilde{p}_1, \tilde{\gamma}(s_1)]$. This concatenation is a geodesic by the choice of α and the construction of the cone. Otherwise, repeat the procedure at $\tilde{p}_2$ and so on.

We will need to repeat this procedure at most $\frac{T + \frac{\theta_0}{2\eta}}{\ell_0}$ times. We extend the beginning of $\tilde{\gamma}_e$ constructed here with $[\tilde{\gamma}(s_1), \tilde{\gamma}(t + s_2)]$ and then extend beyond $\tilde{\gamma}(t + s_2)$ (if needed) similarly to the procedure at $\tilde{\gamma}(s_1)$. Since the turning angles at each cone point are at least π, we obtain a saddle connection path $\tilde{\gamma}_e$ with the desired property for $T_1 = T + \frac{\theta_0}{2\eta} + T_0$. ☐

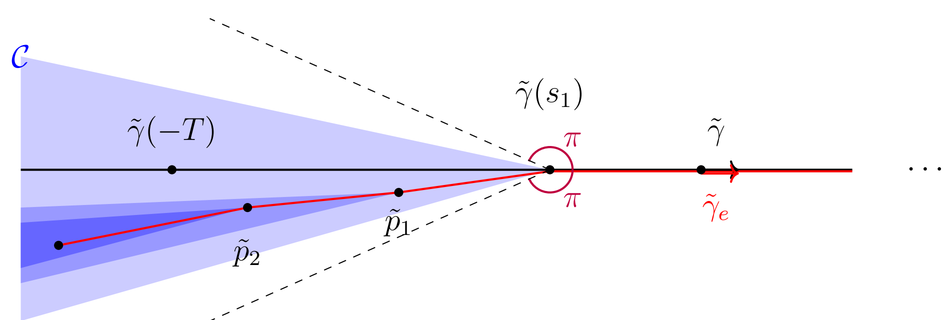

FIGURE 2. The construction of γ_e in Lemma 4.2 around the left endpoint of γ. Shaded in blue is the sequence of α-cones featured in the proof.

Lemma 4.3. *(Compare with Lemma 3.9 in* [Dan11]*) Let $A = \min\{\mathcal{L}(p) - 2\pi | p \in Con\}$ and $N = [\frac{4\pi}{A}] + 3$. Let $q \in Con$. Then there exist N saddle connections $s_1, s_2, \ldots, s_N$ emanating from q with the following property:*

For any local geodesic c with endpoint q, the concatenation of c with at least one s_i is also a local geodesic.

Proof. We have $\mathcal{L}(q) = 2\pi + \alpha \geq 2\pi + A$. Divide the space of directions at q into intervals of size no more than $\frac{\alpha}{2}$; at most $\lceil \frac{2\pi+\alpha}{\alpha/2} \rceil \leq N$ intervals are needed. Using Lemma 4.1, pick a saddle connection emanating from q with direction in each of these intervals. These are the s_i.

The concatenation of c and some saddle connection s_i is a geodesic if and only if s_i lies outside of the π-cone of directions at q with center c. The complement of this cone in the space of directions at q is an interval of size $\mathcal{L}(q) - 2\pi = \alpha$ and must therefore fully contain one of our $\frac{\alpha}{2}$-size intervals. The s_i chosen in this interval geodesically continues c as desired. □

Proposition 4.4. *(Compare with Proposition 3.2 in* [Dan11]*) There exists a constant $C(S) > 0$ so that the following holds:*

For any two parametrized saddle connections s, s' on S there exists a geodesic c which first passes through s and eventually passes through s' and which is of length at most $C(S) + \ell(s) + \ell(s')$.

Proof. The proof follows the proof of Proposition 3.2 in [Dan11] by replacing [Dan11, Lemma 3.9] by Lemma 4.3. □

Using Lemma 4.2 and Proposition 4.4, we obtain the weak specification property on $\mathcal{G}(\eta)$ at all scales.

Corollary 4.5. *(Weak specification) For all $\delta > 0$ there exists $T = T(\eta, \delta, S) > 0$ such that for all $(\gamma_1, t_1), \ldots, (\gamma_k, t_k) \in \mathcal{G}(\eta)$ there exist $0 = s_1 < s_2 < \ldots < s_k$ and a geodesic γ on S such that for all $i = 1, \ldots, k$ we have $s_{i+1} - (s_i + t_i) \in [0, T]$ and $d_{GS}(g_s(\gamma_i), g_s(g_{s_i}(\gamma)) < \delta$ for all $s \in [0, t_i]$.*

We can take $T = 2\max\{s, T_1\} + C(S)$ where T_1 is as in Lemma 4.2 and $C(S)$ is as in Proposition 4.4.

5. $\mathcal{G}(\eta)$ has strong specification (at all scales)

The goal of this section is to prove Proposition 5.6.

As η is fixed throughout, we write $\mathcal{G} := \mathcal{G}(\eta)$.

Lemma 5.1. *If $G \subset \mathbb{R}^{\geq 0} \not\subset c\mathbb{N}$ for all $c > 0$, then for all $\delta > 0$, there exist $x, y \in G$ and $n, m \in \mathbb{N}$ such that $0 < nx - my < \delta$.*

Proof. Let x denote the smallest non-zero element of G, which exists, as otherwise we are immediately done. Now, there are three cases.

First, assume there exists $y \in G$ such that $\frac{y}{x} \notin \mathbb{Q}$. Now take $q \in \mathbb{N}$ large enough so that $\frac{x}{q} < \delta$, and so that there is $p \in \mathbb{N}$ with $|\frac{y}{x} - \frac{p}{q}| < \frac{1}{q^2}$ by Dirichlet's theorem. Then, this implies that

$$|qy - px| < \frac{x}{q} < \delta.$$

In the second case, suppose that for all $y \in G$, $\frac{y}{x}$ is rational, and when written in lowest terms, the denominators can be arbitrarily large. Then, take n such that $\frac{x}{n} < \delta$ and $y \in G$ with $\frac{y}{x} = \frac{p}{q}$ in lowest terms for some $q > n$. Then, as p is invertible in $\mathbb{Z}/q\mathbb{Z}$, we can take m to be a positive integer such that $mp = 1 \pmod q$. It follows that

$$\left| \frac{mp-1}{q} x - my \right| = \frac{x}{q} < \delta.$$

Finally, in the third case, $\frac{y}{x}$ is always rational, but with denominators bounded above by M. Then, $G \subset \frac{x}{M!}\mathbb{N}$, a contradiction. □

Lemma 5.2. *Suppose $x > y > 0$ and $x - y = \delta$. Then, there exists $T > 0$ such that for all $\tau \geq T$ and all $n \in \mathbb{N} \cup \{0\}$, there exists $m_1, m_2 \in \mathbb{N}$ such that $\tau + n\delta \leq m_1 x + m_2 y \leq \tau + (n+1)\delta$.*

Proof. Fix C such that $C > \frac{y}{\delta} + 2$. We claim that $T = \max\{Cy, 1\}$. Fix $\tau \geq T$. Now, let $n \in \mathbb{N} \cup \{0\}$. Fix k_1 to be the largest integer such that $k_1 y \leq \tau + n\delta$ and then choose k_2 to be the smallest integer (positive) such that $k_1 y + k_2\delta \geq \tau + n\delta$. Therefore, we see that $k_2 x + (k_1 - k_2)y = k_1 y + k_2 \delta$, and so

$$\tau + n\delta \leq k_2 x + (k_1 - k_2)y \leq \tau + (n+1)\delta.$$

Observe that by construction,

$$k_1 y + (k_2 - 1)\delta < \tau + n\delta < k_1 y + y,$$

and consequently, $k_2 < \frac{y}{\delta} + 1$. Therefore, by our choice of τ, we see

$$k_1 > \frac{\tau + n\delta - y}{y} > \frac{Cy - y}{y} > \frac{y}{\delta} + 1.$$

Thus, $k_1 - k_2 > 0$, and we are done. □

We will need the following result of Ricks, where we explain the necessary terminology in the course of applying it:

Theorem 5.3. [Ric17, Theorem 4] *Let X be a proper, geodesically complete CAT(0) space under a proper, cocompact, isometric action by a group Γ with a rank one element, and suppose X is not isometric to the real line. Then, the length spectrum is arithmetic if and only if there is some $c > 0$ such that X is isometric to a tree with all edge lengths in $c\mathbb{Z}$.*

Proposition 5.4. *Given $\delta > 0$, there exist two closed saddle connection paths γ, ξ such that $0 < |\ell(\gamma) - \ell(\xi)| < \delta$.*

Proof. This follows for translation surfaces by combining Lemma 5.1 with §6 of [CP20] (see hypothesis (T3) and the discussion following [CP20, Proposition 6.9]).

For general flat surfaces with conical points, this follows from Theorem 5.3. We outline the reasoning as follows. We say that $\gamma \in \Gamma$ is rank one if there exists a geodesic η such that $\gamma\eta = g_t\eta$ for some $t > 0$ and η does not bound a flat half strip. The existence of this follows from the existence of a closed geodesic which turns with angle greater than π at some cone point. Now, the universal cover of a flat surface with cone points is not isometric to a tree with edge lengths in $c\mathbb{Z}$, and so it follows that the length spectrum is not arithmetic. The length spectrum is the collection of lengths of hyperbolic isometries in Γ, which is precisely the set of lengths of closed geodesics, which by Lemma 2.14 is the set of lengths of closed saddle connection paths. We can now apply Lemma 5.1.

□

Proposition 5.5. *For all $\delta > 0$, there exists $\tau > 0$ and $\delta' < \delta$ such that for any two saddle connections s, s' and any $n \in \mathbb{N} \cup \{0\}$, there exists a geodesic ξ_n which begins at s and ends at s' with length in $[\ell(s) + \ell(s') + \tau + n\delta', \ell(s) + \ell(s') + \tau + (n+1)\delta']$.*

Proof. Fix $\delta > 0$ and take γ_1, γ_2 to be closed geodesics such that $0 < |\ell(\gamma_1) - \ell(\gamma_2)| = \delta' < \delta$, which exist by Proposition 5.4. Now take $\tau = 3C(S) + T$, where $C(S)$ is from Lemma 4.4 and T is from Lemma 5.2 applied for $\ell(\gamma_1)$ and $\ell(\gamma_2)$.

Consider two saddle connections s and s' and apply Proposition 4.4 three times to connect, in sequence, s to γ_1 to γ_2 to s' with the geodesic ξ. Furthermore, $\ell(\xi) = L + \ell(s) + \ell(\gamma_1) + \ell(\gamma_2) + \ell(s')$ and $L \leq 3C(S)$. Because the γ_i are closed geodesics, there is a geodesic ξ_{k_1,k_2} which follows the exact path of ξ except that it loops around γ_i a total of k_i times. In other words, $\ell(\xi_{k_1,k_2}) = \ell(\xi) + (k_1 - 1)\ell(\gamma_1) + (k_2 - 1)\ell(\gamma_2)$. Now let $n \in \mathbb{N}$, and take k_1, k_2 such that

$$k_1\ell(\gamma_1) + k_2\ell(\gamma_2) \in [T + (3C(S) - L) + n\delta', T + (3C(S) - L) + (n+1)\delta'].$$

Then $\xi_n := \xi_{k_1,k_2}$ satisfies our desired property. □

Proposition 5.6. *The collection of orbit segments $\mathcal{G}$ has strong specification at all scales.*

Proof. By adapting our proof of weak specification to use Proposition 5.5, we now have the following version of the specification property. For all $\varepsilon, \delta > 0$ there exists $T > 0$ and $\delta' \in (0, \delta)$ such that given any collection of orbit segments $\{(\gamma_i, t_i)\}_{i=1}^n$ and $\mathbf{k} = (k_1, \cdots, k_{n-1}) \in (\mathbb{N} \cup \{0\})^{n-1}$, there exists a geodesic $\xi_{\mathbf{k}}$ such that for $1 \leq i \leq n$, $d_{GS}(g_u g_{s_i} \xi_{\mathbf{k}}, g_u \gamma_i) \leq \varepsilon$ for all $u \in [0, t_i]$, where $s_i = \sum_{j=1}^{i-1}(t_j + T + c_j)$ for some $c_j \in [k_j\delta', (k_j+1)\delta']$. Furthermore, the c_j are independent of all but the j-th coordinate of $\mathbf{k}$.

This implies the strong specification property. We give a brief overview of the idea before giving the details. We will use the property discussed above to shadow our good orbit segments at a scale of $\frac{\varepsilon}{2}$, with the ability to control how long it takes to get from one orbit segment up to a time difference of $\frac{\varepsilon}{4}$. Therefore, if our shadowing geodesic ever gets too far ahead of the orbit segments due to this variance, we can take slightly longer.

Let $\varepsilon > 0$, and choose our specification constant $\tau := T + \frac{\varepsilon}{4}$, where T is chosen as above for the constants $\frac{\varepsilon}{2}$ and $\frac{\varepsilon}{4}$. Now let $\{(\gamma_i, t_i)\}_{i=1}^n$ be a collection of good orbit segments. Take our shadowing geodesic segment to be $\xi_{\mathbf{k}}$ where we choose k_j successively, so that $k_1 = 0$ and $k_{j+1} = \lceil \frac{\varepsilon}{4\delta'} \rceil$ if $j\frac{\varepsilon}{4} - \sum_{i \leq j} c_i > \frac{\varepsilon}{4}$, and 0 otherwise, as this ensures $\left| \sum_{j=1}^{i-1} \frac{\varepsilon}{4} - c_j \right| < \frac{\varepsilon}{2}$ for all $i \leq n$. (See Figure 3.) Taking $\xi := \xi_{\mathbf{k}}$, observe that for $1 \leq i \leq n$

$$d_{GS}(g_u g_{\sum_{j=1}^{i-1}(t_j + T + \varepsilon/4)} \xi, g_u \gamma_i) \leq d_{GS}(g_u g_{\sum_{j=1}^{i-1}(t_j + T + c_j)} \xi, g_u \gamma_i) + \frac{\varepsilon}{2} \leq \varepsilon$$

for all $u \in [0, t_i]$ because the geodesic flow moves at unit speed. This completes our proof. □

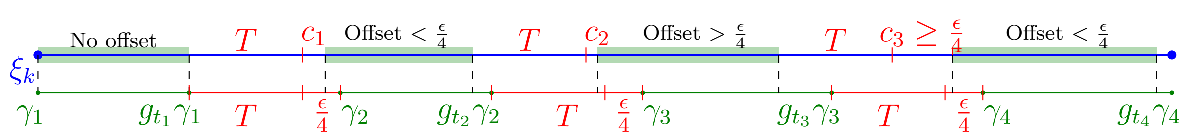

FIGURE 3. A schematic for the arrangement of orbit segments in the proof of Proposition 5.6.

We close this section by recording a simple technical modification of Proposition 5.6 which we will need when we apply weak specification in Section 7.

Definition 5.7. *Let $\tau > 0$ and $\eta > 0$ be given. We denote by $\mathcal{G}^\tau(\eta)$ the set of all orbit segments (γ, t) such that there exist t_1, t_2 with $|t_i| < \tau$ such that $(g_{t_1}\gamma, t - t_1 + t_2) \in \mathcal{G}(\eta)$. That is, these are segments which lie in $\mathcal{G}(\eta)$ after making some bounded change to their endpoints.*

Corollary 5.8. *Specification as in Proposition 5.6 holds for $\mathcal{G}^\tau(\eta)$, with the constant T depending on τ in addition to the parameters listed in Proposition 5.6.*

Proof. This is a simple exercise using Corollary 4.5 and uniform continuity of the geodesic flow. We give the idea of the proof. Let $\{(\gamma_i, t_i)\}_{i=1}^n \subset \mathcal{G}^\tau(\eta)$ we wish to shadow at scale ϵ. Consequently, this leads to a collection $\{(g_{s_i}\gamma_i, t_i')\}_{i=1}^n \subset \mathcal{G}(\eta)$, where $0 \leq s_i \leq \tau$ and $t_i - \tau \leq t_i' \leq t_i$ which we can shadow at any scale. We choose our new shadowing scale δ so that if $d_{GS}(\gamma, \xi) < \delta$, then $d_{GS}(g_t\gamma, g_t\xi) < \varepsilon$ for $t \in [-\tau, \tau]$. Any sequence which δ-shadows $\{(g_{s_i}\gamma_{t_i}, t_i')\}$ must then ε-shadow our desired collection $\{(\gamma_i, t_i)\}$. □

6. $\mathcal{G}(\eta)$ has the Bowen property

In this section we establish the Bowen property (see Definition 2.6). To do so, we analyze orbits that stay close to a good orbit segment for some time. This description will allow us to effectively bound the difference of ergodic averages along these orbits.

Proposition 6.1. *For all $\eta > 0$, for all sufficiently small $\varepsilon > 0$ (dependent on η), and for any $(\gamma, t) \in \mathcal{G}(\eta)$ with $t > 2\frac{\theta_0}{2\eta}$, we have*

$$B_t(\gamma, \varepsilon) \subset C_{2\varepsilon, \frac{\theta_0}{2\eta}}(\gamma, t),$$

where

$$B_t(\gamma, \varepsilon) = \{\xi \in GS : d_{GS}(g_u\gamma, g_u\xi) < \varepsilon \text{ for all } u \in [0, t]\}$$

and

$$C_{2\varepsilon, \frac{\theta_0}{2\eta}}(\gamma, t) = \left\{\xi : \exists |r| \leq 2\varepsilon \text{ such that } g_r\xi(u) = \gamma(u) \text{ for all } u \in [\frac{\theta_0}{2\eta}, t - \frac{\theta_0}{2\eta}]\right\}.$$

Proof. Fix $\eta > 0$, and recall Corollary 3.9. Now choose $\varepsilon > 0$ such that for any cone point, the ball of radius $2\varepsilon e^{2(\frac{\theta_0}{2\eta} + s)}$ with center at distance s from the cone point and located on the axis of a cone generated by the vectors that make angle $\frac{s\eta}{4}$ with its axis is contained in the cone (recall that $s > 0$ was chosen so that $2s < \ell_0$).

Let $(\gamma_1, t) \in \mathcal{G}(\eta)$ with $t > \frac{\theta_0}{\eta}$ be arbitrary. By Corollary 3.9, there exists $t_0 \in [-\frac{\theta_0}{2\eta}, \frac{\theta_0}{2\eta}]$ such that $\gamma_1(t_0) \in Con$ and $|\theta(\gamma_1, t_0)| - \pi \geq s\eta$. Similarly, there exists $t_1 \in [-\frac{\theta_0}{2\eta}, \frac{\theta_0}{2\eta}]$ such that $\gamma_1(t + t_1) \in Con$ and $|\theta(\gamma_1, t + t_1)| - \pi \geq s\eta$.

Now consider $\gamma_2 \in B_t(\gamma_1, \varepsilon)$. Taking t_0 and t_1 as above, by Lemmas 2.8 and 2.11,

$$d_S(\gamma_1(t_0 - s), \gamma_2(t_0 - s)) \leq 2d_{GS}(g_{t_0 - s}\gamma_1, g_{t_0 - s}\gamma_2) \leq 2d_{GS}(\gamma_1, \gamma_2)e^{2|\frac{\theta_0}{2\eta} - s|} \leq 2\varepsilon e^{2(\frac{\theta_0}{2\eta} + s)}, \quad (6.1)$$

and

$$d_S(\gamma_1(t+t_1+s), \gamma_2(t+t_1+s)) \leq 2d_{GS}(g_{t_1+s}g_t\gamma_1, g_{t_1+s}g_t\gamma_2) \leq 2d_{GS}(g_t\gamma_1, g_t\gamma_2)e^{2|t_1+s|} \leq 2\varepsilon e^{2(\frac{\theta_0}{2\eta} + s)}. \quad (6.2)$$

Now let $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ be the lifts of γ_1 and γ_2 to $G\tilde{S}$ so that $d_{G\tilde{S}}(\tilde{\gamma_1}, \tilde{\gamma_2}) = d_{GS}(\gamma_1, \gamma_2)$.

Let $B_1 = B(\tilde{\gamma}_1(t_0 - s), 2\varepsilon e^{2(\frac{\theta_0}{2\eta}+s)})$ and $B_2 = B(\tilde{\gamma}_1(t + t_1 + s), 2\varepsilon e^{2(\frac{\theta_0}{2\eta}+s)})$. Then, by (6.1) and (6.2), γ_2 intersects B_1 and B_2. Since $|\theta(\gamma_1, t_0)| - \pi \geq s\eta$ and $|\theta(\gamma_1, t + t_1)| - \pi \geq s\eta$, by the choice of ε and the fact that any two points in a CAT(0)-space are connected by a unique geodesic segment, $\tilde{\gamma_2}$ contains $\tilde{\gamma}_1[t_0, t + t_1]$. Moreover, since $d_{G\tilde{S}}(g_{t_0+s}\tilde{\gamma}_1, g_{t_0+s}\tilde{\gamma}_2) \leq \varepsilon$ and $0 \leq t_0 + s \leq 2s < t$, it follows that $d_S(\tilde{\gamma}_1(t_0 + s), \tilde{\gamma}_2(t_0 + s)) \leq 2\varepsilon$. Thus, there exists r such that $|r| \leq 2\varepsilon$ and $g_r\gamma_2(u) = \gamma_1(u)$ for $u \in [t_0, t + t_1]$. Since $t_0 \leq \frac{\theta_0}{2\eta}$ and $t_1 \geq -\frac{\theta_0}{2\eta}$, we have completed our proof.

□

Proposition 6.2. *For all $\varepsilon, s > 0$ and α-Hölder continuous functions ϕ, there exists $K > 0$ such that for all geodesic segments (γ_1, t) with $t > 2\frac{\theta_0}{2\eta}$, given any $\gamma_2 \in C_{2\varepsilon,s}(\gamma_1, t)$, we have*

$$\left| \int_0^t \phi(g_r\gamma_1) - \phi(g_r\gamma_2)\, dr \right| \leq K.$$

Proof. Let R be the time-shift in the definition of $C_{2\varepsilon,s}(\gamma_1, t)$, so that $g_R\gamma_2(r) = \gamma_1(r)$ for $r \in [s, t - s]$. We see that

$$\begin{aligned}\left| \int_0^t \phi(g_r\gamma_1) - \phi(g_r\gamma_2)\, dr \right| &\leq \left| \int_0^t \phi(g_r\gamma_1)\, dr - \int_{-R}^{t-R} \phi(g_r(g_R\gamma_2))\, dr \right| \\ &\leq \left| \int_s^{t-s} \phi(g_r\gamma_1) - \phi(g_r(g_R\gamma_2))\, dr \right| + (4s + 2|R|)\|\phi\|.\end{aligned}$$

Since $\gamma_1 = g_R\gamma_2$ on $[s, t - s]$, by Lemma 2.10, we have for all $r \in [s, t - s]$

$$d_{GS}(g_r\gamma_1, g_r(g_R)\gamma_2) \leq e^{-2\min\{|r-s|,|r-(t-s)|\}}.$$

Thus, we obtain

$$\begin{aligned}\left| \int_s^{t-s} \phi(g_r\gamma_1) - \phi(g_r(g_R\gamma_2))\, dr \right| &\leq \int_s^{t-s} (d_{GS}(g_r\gamma_1, g_r(g_R\gamma_2)))^\alpha\, dr \\ &\leq \int_s^{\frac{t}{2}} e^{-2\alpha(r-s)}\, dr + \int_{\frac{t}{2}}^{t-s} e^{-2\alpha((t-s)-r)}\, dr \\ &= \frac{1}{\alpha}(1 - e^{-\alpha(t-2s)}) \\ &\leq \frac{1}{\alpha}.\end{aligned}$$

As a result, since $|R| < 2\varepsilon$, we have

$$\left| \int_0^t \phi(g_r\gamma_1) - \phi(g_r\gamma_2)\, dr \right| \leq \frac{1}{\alpha} + (4s + 4\varepsilon)\|\phi\|.$$

□

Corollary 6.3. *For all $\eta > 0$, there exists $\varepsilon > 0$ such that $\mathcal{G}(\eta)$ has the Bowen property at scale ε.*

Proof. Fix $\eta > 0$. Then, choose $\varepsilon > 0$ sufficiently small to apply the previous propositions. Then, we can take the constant for the Bowen property to be $\max\{K, 2\frac{\theta_0}{\eta}\|\phi\|\}$, where K is from the previous proposition. Then, the previous proposition gives the desired bound for orbit segments of length at least $\frac{\theta_0}{\eta}$, and the triangle inequality gives the desired bound for any shorter orbit segments. □

7. Establishing the Pressure Gap

In this section, we prove the pressure gap condition of [BCFT18] for certain potentials. We then show that this pressure gap holds in the product space as well. See also the survey by Climenhaga and Thompson [CT20, Section 14].

First, we prove the following theorem.

Theorem 7.1. *Let ϕ is a continuous potential that is locally constant on a neighborhood of* Sing*. Then, $P(\mathrm{Sing}, \phi) < P(\phi)$.*

Furthermore, we use the above theorem to note that a pressure gap also holds for functions that are nearly constant. See Corollary 7.6.

Our argument for Theorem 7.1 closely follows that in §8 of [BCFT18]. The different geometry in our situation calls for somewhat different arguments in Proposition 7.3 and Lemma 7.4, which we present here in full. After these are proved, the argument hews closely to [BCFT18]. We present the main steps of the argument, filling in the few details where a modification is necessary for the present situation.

For any $\eta > 0$, we let

$$\mathrm{Reg}(\eta) = \{\gamma : \lambda(\gamma) \geq \eta\}.$$

We also note

Lemma 7.2. *Let $[p, q]$ be any singular geodesic segment. That is, $[p, q]$ is a geodesic segment from p to q such that the turning angle τ at any cone points it encounters is always of magnitude π. Then $[p, q]$ can be extended to a complete geodesic $\gamma \in \mathrm{Sing}$.*

Proof. The extension is accomplished by following the geodesic trajectory established by $[p, q]$ and, whenever a cone point is encountered, continuing the extension so that a turning angle of π or $-\pi$ is made. □

The first step in the dynamical argument for a pressure gap is the following technical Proposition, which allows us to find a regular geodesic which is close to any connected component of the δ-neighborhood of the singular set.

Proposition 7.3. *Let $\delta > 0$ and $0 < \eta < \frac{\eta_0}{2s}$ be given, where η_0 is defined in Lemma 2.12. Then there exists $L > 0$ and a family of maps $\Pi_t : \mathrm{Sing} \to \mathrm{Reg}(\eta)$ such that for all $t > 3L$ and for all $\gamma \in \mathrm{Sing}$, if we write $c = \Pi_t(\gamma)$ then the following are true:*

(a) *$c, g_{t+\tau}c \in \mathrm{Reg}(\eta)$ for some $|\tau| < 4d_0$;*
(b) *$d_{GS}(g_s c, \mathrm{Sing}) < \delta$ for all $s \in [L, t - L]$;*
(c) *for all $s \in [L, t - L]$, $g_s c$ and γ lie in the same connected component of $B(\mathrm{Sing}, \delta)$, the δ-neighborhood of* Sing.

Remark. *The above proposition should be compared with* [BCFT18, Theorem 8.1]*, although we have made two slight adjustments for our situation. First, we cannot guarantee that $g_t c \in \mathrm{Reg}(\eta)$, but only that $g_{t+\tau}c \in \mathrm{Reg}(\eta)$ with uniform control on $|\tau|$. Second, we prove* *our result for all $t > 3L$, instead of $2L$. These result in trivial changes to subsequent estimates in* [BCFT18]*'s argument.*

Proof of Proposition 7.3. We work in $\tilde{S}$. If the result holds there, it does for S, as our arguments will clearly be invariant under Γ.

We begin with a few remarks.

First, to ensure $c, g_{t+\tau}c \in \mathrm{Reg}(\eta)$ it is enough to ensure that at some point in each time interval $[-s, s]$ and $[t+\tau-s, t+\tau+s]$, c hits a cone point and turns with angle $\geq \pi + \eta s$. Since $\eta < \frac{\eta_0}{2s}$, we see that any geodesic segment hitting a cone point in the desired time intervals can be extended so that it satisfies this turning angle condition – every cone point has at least η_0 excess angle allowing for extensions with turning angle $\geq \eta_0/2$.

Second, given the geodesic $\gamma \in \mathrm{Sing}$, draw geodesic segments of length d_0 (see Lemma 2.12) perpendicular to γ, to the right of γ (with respect to its orientation) and based at $\gamma(-d_0)$, $\gamma(d_0)$, $\gamma(t-d_0)$, and $\gamma(t+d_0)$. Connect the endpoints of the first two segments to form a geodesic quadrilateral in $\tilde{S}$; do the same with the latter two segments. (See the red quadrilaterals in Figure 4; we have assumed, without loss of generality, that L will be picked much larger than d_0. If any of the segment perpendicular to γ hit cone points, choose the geodesic extensions which make angle π to the inside of the quadrilateral.) By Lemma 2.12(a), each of these quadrilaterals must contain at least one cone point. Let ζ_1 and ζ_2 be the cone points in these quadrilaterals which are closest to the image of γ (any ties may be broken in an arbitrary fashion).

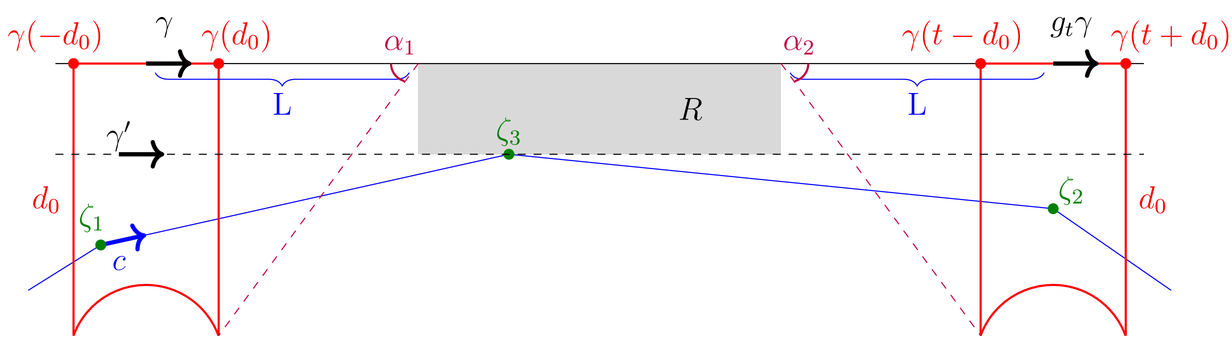

FIGURE 4. The construction of $c = \Pi_t(\gamma)$ in Proposition 7.3.

Take the geodesic segment connecting ζ_1 and ζ_2, and extend it to a complete geodesic c so that the turning angles at ζ_1 and ζ_2 are $\geq \pi + \eta s$, following our first remark at the start of the proof. Parametrize c so that $c(0) = \zeta_1$. Then, $\zeta_2 = c(t+\tau)$ for some $|\tau| \leq 4d_0$; this can be seen from the distance minimizing properties of geodesics and a quick application of the triangle inequality. We declare $\Pi_t(\gamma) = c$. By construction, c and $g_{t+\tau}c$ are in $\mathrm{Reg}(\eta)$.

We now need to pick L so that conditions (b) and (c) of the proposition hold for all $t > 3L$. Consider, as in Figure 4, the angles α_1 and α_2 formed at $\gamma(L)$ and $\gamma(t-L)$ between the image of γ and the geodesic segments to the corners of the red quadrilaterals shown. Since d_0 is constant, by increasing L we can make both α_i as small as we like.

Now, for any choice of L, consider the maximal closed, flat rectangle R with one side equal to $[\gamma(L), \gamma(t-L)]$ lying to the right of γ. (R has width at least L, but note that R may have zero or infinite height in Figure 4.) We claim that, for a sufficiently large choice of L,

$c \cap R \neq \emptyset$. To see this, choose L so large that $\alpha_1, \alpha_2 < \eta_0/2$. Now if R has infinite height, the result is clearly true. If not, it must be that the side of R parallel to its side along γ contains a cone point ζ_3, as shown in Figure 4. If $[\zeta_1, \zeta_2]$ passes strictly above ζ_3 in Figure 4, we are again done, so suppose this is not the case. It is easy to see that the angles between the bottom side of R and the segments $[\zeta_1, \zeta_3]$ and $[\zeta_3, \zeta_2]$ are less than the angles α_1 and α_2, respectively, hence less than $\eta_0/2$ by our choice of L. Since the cone point ζ_3 has at least η_0 excess angle, it is easily verified that the concatenation of the geodesic segments $[\zeta_1, \zeta_3]$ and $[\zeta_3, \zeta_2]$ has angles $\geq \pi$ on both sides as it passes through ζ_3. Hence, it is *a* geodesic, and therefore *the* geodesic segment between ζ_1 and ζ_2. This proves the claim.

Let $\gamma' \in \text{Sing}$ be a singular geodesic created in the following manner.

- If c intersects both vertical sides of R, let γ' be an extension of $c \cap R$ to an element of Sing using Lemma 7.2. Parametrize γ' so that on their intersection within R, $c(t) = \gamma'(t)$.
- Otherwise, c must intersect the bottom side of R. Let γ' be an extension of the bottom side of R to an element of Sing (this is the case shown in Figure 4). Again, parametrize γ' so that where they intersect the parametrizations of γ' and c agree.

In either case, there exists some $t^* \in [L - d_0, t - L + 2d_0]$ such that $c(t^*) = \gamma'(t^*)$. Let $r_1 > 0$ be the smallest value such that $c(t^* - r_1)$ intersects the left-hand red quadrangle; let $r_2 > 0$ be the smallest value such that $c(t^* + r_2)$ hits the right red quadrangle. It is clear from our picture that $d_{\tilde{S}}(c(t^* - r_1), \gamma'(t^* - r_1)) < 2d_0$ and $d_{\tilde{S}}(c(t^* + r_2), \gamma'(t^* + r_1)) < 2d_0$. By the convexity of the distance function between geodesics, it must therefore be the case that

$$d_{\tilde{S}}(c(t^* + r), \gamma'(t^* + r)) \leq Ar \qquad \text{for all } \tau \in [-r_1, r_2]$$

where $A = \max\{2d_0/r_1, 2d_0/r_2\}$. Clearly, by further increasing L, we can make r_1 and r_2 as large as we like, and hence A as small as we like. Therefore, by choosing L sufficiently large, we can ensure that for all $s \in [L - T(\delta), t - L + T(\delta)]$, where $T(\delta)$ is provided by Lemma 2.9, $d_{\tilde{S}}(c(s), \gamma'(s)) < \delta/2$. Then, Lemma 2.9 tells us that $d_{GS}(g_s c, g_s \gamma') < \delta$ for all $t \in [L, t - L]$ as was desired for statement (b).

It remains to prove statement (c), which we can do by proving that γ' is in the same connected component of $B(\text{Sing}, \delta)$ as γ. The geodesic flow gives continuous paths through Sing, so γ and $g_{\frac{t}{2}}\gamma$ are in the same connected component of $B(\text{Sing}, \delta)$. Shifting the geodesic segment $\gamma \cap R$ down through R gives a continuous family of geodesic segments each of which can be extended (using Lemma 7.2) to geodesics in Sing. By again choosing L large (greater than $2T(\delta)$ in this case) and using Lemma 2.9 we can easily find a finite sequence $g_{\frac{t}{2}}\gamma = \gamma_1, \ldots, \gamma_n$ of these geodesics in Sing such that $d_{GS}(\gamma_i, \gamma_{i+1}) < \delta$, and such that γ_n and γ' intersect in R. Therefore, γ_n is still in the original connected component of $B(\text{Sing}, \delta)$. Finally, if $\gamma_n(0) = \gamma'(t')$, then $g_{t'}\gamma'$ and γ_n intersect at time zero and the distance between them grows at most at a slow linear rate again controlled by our choice of L in the same way the distances between c and γ' were controlled. Our previous argument tells us $d_{GS}(\gamma_n, g_{t'}\gamma') < \delta$, and therefore γ' is in the same connected component of $B(\text{Sing}, \delta)$ as γ_n. This completes the proof.

□

The second step in the argument is to prove the following Lemma, which uniformly controls how many geodesics in Sing can have image under Π_t near to a fixed geodesic.

Lemma 7.4 (Compare with Prop. 8.2 in [BCFT18])**.** *For all $\epsilon > 0$, there exists some $C(\epsilon) > 0$ such that if $E_t \subset \mathrm{Sing}$ is a $(t, 2\epsilon)$-separated set for some $t > 3L$, then for any $w \in GS$,*

$$\#\{\gamma \in E_t : d_{GS}(w, \Pi_t(\gamma)) < \epsilon\} \leq C.$$

Proof. It is sufficient, and easier, to prove the result in $G\tilde{S}$.

Let d_0 be as in Lemma 2.12. Fix $w \in G\tilde{S}$, and let $\epsilon > 0$ and $t > 3L$ be given.

First, since the cone points in $\tilde{S}$ are (uniformly) discrete, there exists a $C_1(\epsilon) > 0$ such that the collection $\mathcal{C}_w$ of geodesic segments $c : [0, t+\tau] \to \tilde{S}$ with $|\tau| < 4d_0$ with $c(0), c(t+\tau) \in Con$ and $\int_0^{t+\tau} d_{\tilde{S}}(c(s), w(s))e^{-2|s|}ds < \epsilon$ has cardinality at most C_1. We note that C_1 does not depend on t, only on how many cone points can lie in a ball of fixed radius around a point in $\tilde{S}$.

From the construction of Π_t, it is clear that any $\Pi_t(\gamma)$ which lies within ϵ of w must extend a geodesic segment in $\mathcal{C}_w$.

Second, if for some $\gamma \in E_t$, $\Pi_t(\gamma)$ extends some $c \in \mathcal{C}_w$, it is clear from the construction of Proposition 7.3 that $d_{\tilde{S}}(\gamma(0), c(0))$ and $d_{\tilde{S}}(\gamma(t), c(t+\tau))$ are both at most $2d_0$. We claim that there is a $C_2(\epsilon) > 0$ such that any $(t, 2\epsilon)$-separated subset of geodesics which pass through $B(c(0), d_0)$ at time $t = 0$ and through $B(c(t+\tau), d_0)$ at time t has cardinality $\leq C_2$. This follows from some very rough estimates using the compactness of S. (Note that we do not use anything about the properties of Sing for these estimates.)

- There exists some $C'(\epsilon) > 0$ such that any ball of radius d_0 has an $\epsilon/4$-spanning subset for the metric $d_{\tilde{S}}$ with cardinality $\leq C'$.
- There is some $C''(\epsilon)$ such that for any ball B of radius d_0, there is an $\epsilon/4$ spanning subset of the geodesic rays $\{\gamma : (-\infty, 0] \to \tilde{S}$ such that $\gamma(0) \in B\}$ with respect to the metric $d'_{G\tilde{S}}(\gamma, \gamma') := \int_{-\infty}^0 d_{\tilde{S}}(\gamma(s), \gamma'(s))e^{-2|s|}ds$.
- Similarly, there is an $\epsilon/4$-spanning subset of rays with domain $[0, \infty)$ starting in B with respect to the analogous metric with cardinality at most $C''(\epsilon)$.

We claim that $C_2 = C'^2 C''^2$ satisfies the required property.

Indeed, consider the geodesics in a $(t, 2\epsilon)$-separated set passing through $B(c(0), d_0)$ at time $t = 0$ and through $B(c(t+\tau), d_0)$ at time t. To each γ in such a set, we can associate a negative ray γ_- from the spanning set of rays ending in $B(c(0), d_0)$, a middle geodesic segment γ_0 connecting a point in the spanning sets for $B(c(0), d_0)$ to a point in the spanning set for $B(c(t+\tau), d_0)$, and a positive ray γ_+ from the spanning set of rays beginning in $B(c(t+\tau), d_0)$ such that:

- $\int_{-\infty}^0 d_{\tilde{S}}(\gamma(s), \gamma_-(s))e^{-2|s|}ds < \epsilon/4$,
- $d_{\tilde{S}}(\gamma(s), \gamma_0(s)) < \epsilon/4$ for all $s \in [0, t]$, and
- $\int_0^{\infty} d_{\tilde{S}}(\gamma(t+s), \gamma_+(s))e^{-2|s|}ds < \epsilon/4$.

There are at most $C'^2 C''^2$ possible such choices. Therefore, if there are more than $C'^2 C''^2$ geodesics in our $(t, 2\epsilon)$-separated set, there are at least two, call them γ_1 and γ_2, which have the same associated negative ray, middle segment, and positive ray under this procedure. It is then straightforward to verify that $d_t(\gamma_1, \gamma_2) < 2\epsilon/4 + 2\epsilon/4 + 2\epsilon/4 < 2\epsilon$, a contradiction.

Third, we can combine these two estimates and get the desired result. By the first part of our argument, there are at most C_1 geodesic segments within ϵ of w which could belong to $\Pi_t(\gamma)$. By the second part, each such segment has pre-image of cardinality at most C_2 in any $(t, 2\epsilon)$-separated E_t. Therefore,

$$\#\{\gamma \in E_t : d_{G\tilde{S}}(w, \Pi_t\gamma) < \epsilon\} \leq C_1C_2 =: C$$

finishing the proof. □

The third step in the argument is to prove the following Lemma.

Lemma 7.5 (Lemma 8.4 in [BCFT18])**.** *For sufficiently small $\delta > 0$, there is a $(t, 2\delta)$-separated set E_t in* Sing *such that there is a (t, δ)-separated set $E_t'' \subset \Pi_t(E_t)$ satisfying*

$$\sum_{w \in E_t''} e^{\inf_{u \in B_t(w,\delta)} \int_0^t \phi(g_s u) ds} \geq \beta e^{tP(\mathrm{Sing},\phi)}$$

where $\beta = \frac{1}{C} e^{-6L\|\phi\|}$, and C is as in Lemma 7.4.

Proof. This Lemma and its proof are almost verbatim as in [BCFT18], specifically Lemmas 8.3 and 8.4 and the discussion between them. □

Note that E_t'' is in $\mathcal{G}^\tau$ for $\tau = 4d_0$, using the notation of Corollary 5.8.

The fourth step in the argument is to use specification to string together orbit segments from E_t'' in many different orders so as to produce a large collection of long orbit segments which together produce more pressure than $P(\mathrm{Sing}, \phi)$. In [BCFT18] this is undertaken in Section 8.4, and at this point the argument is entirely dynamical: the partition sum of Lemma 7.5 together with specification completes the argument precisely as in §8.4 of [BCFT18].

With the pressure gap condition for such potentials in hand we briefly note a second class of potentials for which it holds. Proposition 4.7 of [Cal20] notes that if the pressure gap $P(\mathrm{Sing}, \phi) < P(\phi)$ holds for ϕ, then for any function sufficiently close to ϕ (specifically with $2\|\phi - \psi\| < P(\phi) - P(\mathrm{Sing}, \phi)$) and any constant c, $P(\mathrm{Sing}, \psi + c) < P(\psi + c)$. Applying this to the locally constant functions ϕ discussed in this section gives us a further class of potentials with a pressure gap. Applying it with $\phi = 0$ gives us one class of particular note:

Corollary 7.6. *If ψ is a continuous potential with $\|\psi\| < \frac{1}{2}\left(h_{top}(g_t) - h_{top}(g_t|_{\mathrm{Sing}})\right)$, where h_{top} is the topological entropy, then $P(\mathrm{Sing}, \psi) < P(\psi)$.*

8. Equilibrium states are Limits of Weighted Periodic Orbits

We can show that weighted periodic orbits equidistribute to the equilibrium states we have constructed, following a method of [BCFT18]. Throughout this section, we write $\mathcal{G}^\tau := \mathcal{G}^\tau(\eta)$ (see Definition 5.7) as we will work with a fixed η throughout.

Let $\mathrm{Per}_R[Q - \delta, Q]$ be the set of all regular closed geodesics γ with period in $[Q - \delta, Q]$. Note that if $\gamma \in \mathrm{Per}_R[Q - \delta, Q]$, we do not consider $g_u\gamma$ to also be contained in this set for any $u \neq 0$. Given such a γ, write μ_γ for the normalized Lebesgue measure supported on γ, and $\Phi(\gamma) = \int_0^{\ell(\gamma)} \phi(g_u\gamma)\, du$. We consider a weighted sum over all μ_γ

$$\mu_{Q,\delta} = \frac{1}{\Lambda_R(Q, \delta, \phi)} \sum_{\gamma \in \mathrm{Per}_R[Q-\delta,Q]} e^{\Phi(\gamma)} \mu_\gamma,$$

where $\Lambda_R(Q, \delta, \phi) = \sum_{\gamma \in \mathrm{Per}[Q-\delta,Q]} e^{\Phi(\gamma)}$ is our normalizing constant. When $\lim_{Q\to\infty} \frac{1}{Q} \log \Lambda_R(Q, \delta, \phi)$ exists, it can be thought of as the pressure of closed saddle connection paths, and we write it as $P_{R,\delta}(\phi)$.

Theorem 8.1. *We use the notation above. Let ϕ be a Hölder potential and μ be the unique equilibrium state for ϕ. Then, for all $\delta > 0$, we have $\lim_{Q\to\infty} \mu_{Q,\delta} = \mu$.*

Remark. *Note that this provides a way to identify interesting potentials, by considering geometrically relevant ways to weight closed geodesics. For instance, one could potentially try to identify a continuous function that weights γ by the number of conical points it turns at.*

Proof of Theorem 8.1. We have the following proposition, which follows from the proof of Variational Principle found in [Wal82, Theorem 9.10] because $\mathrm{Per}_R[Q-\delta, Q]$ is (Q,ε)-separated for all sufficiently small ε (as will be shown in Proposition 8.5):

Proposition 8.2. *If μ is the unique equilibrium state for ϕ, then for all $\delta > 0$ such that $\lim_{Q\to\infty} \frac{1}{Q}\log\Lambda_R(Q,\delta,\phi) = P(\phi)$, we have $\lim_{Q\to\infty} \mu_{Q,\delta} = \mu$.*

In order to apply this proposition, we need to establish a growth rate for $\Lambda_R(Q,\delta,\phi)$ for all sufficiently small $\delta > 0$, what is done in Propositions 8.4 and 8.5 which are proved below. Those propositions on the growth rate imply that

$$\lim_{Q\to\infty} \frac{1}{Q}\log\Lambda_R(Q,\delta,\phi) = P(\phi).$$

From this, by Proposition 8.2, it follows that $\lim_{Q\to\infty} \mu_{Q,\delta} = \mu$. ☐

First, we show that the growth rate for $\Lambda_R(Q,\delta,\phi)$ is fast enough. In order to do this, we need to be able to approximate $(\gamma, t) \in \mathcal{G}^\tau$ by closed geodesics of a bounded length. This is encapsulated in the following proposition.

Proposition 8.3. *For all $\delta > 0$, there exists T' such that for all $(\gamma, t) \in \mathcal{G}^\tau$ with $t > \frac{\theta_0}{\eta} + 2\tau$, there is some regular closed geodesic ξ with period in $[t+T'-\delta, t+T']$ such that $d_{GS}(g_u\gamma, g_u\xi) < \delta$ for all $u \in [0,t]$.*

Proof. First, we explain how to obtain the statement of the proposition for $(\gamma, t) \in \mathcal{G}$. Let $\delta > 0$, and let T be the specification constant for $\mathcal{G}$ with shadowing scale $\frac{\delta}{8}$ (see Proposition 5.6). Let $(\gamma, t) \in \mathcal{G}$, and let ξ be a geodesic guaranteed by specification which shadows (γ, t) twice in succession, which we will denote from here by γ_1 and γ_2. Now recall from our proof of strong specification (recall that we use Lemma 4.2 which forces regularity of the shadowing geodesic) that there exists a closed interval $I \supset [\frac{\theta_0}{2\eta}, t - \frac{\theta_0}{2\eta}]$ such that ξ contains $\gamma_1(I)$ and $\gamma_2(I)$. In other words, there exists $r_i > 0$ such that $\xi(r_i + r) = \gamma_i(r)$ for all $r \in I$, where $i \in \{1,2\}$. Thus, we can choose ξ to be a closed geodesic, and observe that its length is given by $r_2 - r_1$. Now, since $d_{GS}(g_{\frac{\theta_0}{2\eta}}\xi, g_{\frac{\theta_0}{2\eta}}\gamma_1) \le \frac{\delta}{8}$ by Proposition 5.6, by Lemma 2.8, $d_S(\xi(\frac{\theta_0}{2\eta}), \gamma_1(\frac{\theta_0}{2\eta})) \le \frac{\delta}{4}$. Thus, $|r_1| \le \frac{\delta}{4}$. Similarly for γ_2, we have $d_S(\xi(\frac{\theta_0}{2\eta} + t + T), \gamma_2(\frac{\theta_0}{2\eta})) \le \frac{\delta}{4}$, and so $r_2 \in [T + t - \frac{\delta}{4}, T + t + \frac{\delta}{4}]$. Hence, ξ is a regular closed geodesic with length in $[T+t-\frac{\delta}{2}, T+t+\frac{\delta}{2}]$. Taking $T' = T + \frac{\delta}{2}$, we are done. In order to adapt this argument to $\mathcal{G}^\tau$ for $\tau > 0$, note that we achieve specification for $\mathcal{G}^\tau$ by considering the specification constant for $\mathcal{G}$ at a smaller scale (which depends on τ). (See Corollary 5.8.) ☐

Proposition 8.4. *For all $\delta > 0$ there exists a constant C such that*

$$\Lambda_R(Q,\delta,\phi) \ge \frac{C}{Q} e^{QP(\phi)}$$

for all sufficiently large Q.

The proof of this proposition follows almost exactly the proof of the lower bound in [BCFT18, Proposition 6.4], replacing the use of [BCFT18, Corollary 4.8] with Proposition 8.3. We include it here for completeness.

Proof. First, observe that there exist $C, \varepsilon, \tau > 0$ so that for all $t > 0$

$$\Lambda(\mathcal{G}^\tau, \varepsilon, t) \geq Ce^{tP(\phi)},$$

where $\Lambda(\mathcal{G}^\tau, \varepsilon, t) = \sup \left\{ \sum_{\gamma \in E} e^{\int_0^t \phi(g_u\gamma)\,du} \mid E \subset \{\gamma \in GS \mid (\gamma, t) \in \mathcal{G}^\tau\} \text{ is } (t,\varepsilon)\text{-separated} \right\}$. This follows from [CT16, Lemma 4.12], and simply means that for all t, there exists a (t, ε)-separated set E_t such that

$$\sum_{\gamma \in E_t} \exp\left(\int_0^t \phi(g_u\gamma)\,du\right) \geq Ce^{tP(\phi)}.$$

Now, choose $\rho < \frac{\varepsilon}{3}$ small enough that the Bowen property holds on $\mathcal{G}^\tau$ (this follows immediately from the fact that $\mathcal{G}$ has the Bowen property). Then, by Proposition 8.3, when $t > \frac{\theta_0}{\eta} + 2\tau$, there exists $T' > 0$ so that there is an injective mapping from E_t to a set P_t of regular closed geodesics with periods in $[t + T' - \delta, t + T']$, i.e. for any $\xi \in P_t$ there exists $u \in [t + T' - \delta, t + T']$ such that $g_u\xi = \xi$. In particular, for all $\gamma \in E_t$, there exists $\xi \in P_t$ so that $d_{GS}(g_u\xi, g_u\gamma) \leq \rho$ for all $u \in [0, t]$. Because the mapping is injective and ϕ has the Bowen property at scale ρ on $\mathcal{G}^\tau$, it follows that

$$\sum_{\xi \in P_t} \exp\left(\int_0^t \phi(g_u\xi)\,du\right) \geq Ce^{-K}e^{tP(\phi)}$$

for some constant K independent of t. Now, writing $\Phi(\xi) = \int_0^{\ell(\xi)} \phi(g_u\xi)\,du$, we can then write

$$\sum_{\xi \in P_t} \exp(\Phi(\xi)) \geq \sum_{\xi \in P_t} \exp\left(\int_0^t \phi(g_u\xi)\,du - T'\|\phi\|\right) \geq Ce^{-(K+T'\|\phi\|)}e^{tP(\phi)}.$$

At this point, we can almost relate this to $\Lambda_R(Q, \delta, \phi)$. However, there is a possibility that $\xi_1, \xi_2 \in P_t$ both represent the same closed geodesic path, i.e., there exists u so that $g_u\xi_1 = \xi_2$. As P_t is (t, ρ)-separated and $d_{GS}(\eta, g_u\eta) = u$, there are at most $\frac{t+T'}{\rho}$ such repetitions. Hence, if $Q \geq T$, by setting $Q = t + T'$, we have

$$\Lambda_R(Q, \delta, \phi) \geq \left(\frac{\rho}{Q}\right) Ce^{-K}e^{-T'(\|\phi\|+P(\phi))}e^{QP(\phi)}.$$

□

In order to see that the growth rate is not too large, we appeal to the Flat Strip theorem.

Proposition 8.5. *For all $\delta > 0$ there exists a constant $C > 0$ such that*

$$\Lambda_R(Q, \delta, \phi) \leq Ce^{\delta\|\phi\|}e^{QP(\phi)}$$

for all sufficiently large Q.

Proof. First, we claim that $\mathrm{Per}_R[Q-\delta, Q]$ is (Q,ε)-separated for all ε sufficiently small. Consider $\gamma_1, \gamma_2 \in \mathrm{Per}_R[Q-\delta, Q]$. Then, if $d_{GS}(g_t\gamma_1, g_t\gamma_2) < \varepsilon$ for all $t \in [0,Q]$, then $d_S(\gamma_1(t), \gamma_2(t)) < 2\varepsilon$ for all $t \in [0,Q]$. Therefore, there are lifts $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ which must be distinct by the definition of $\mathrm{Per}_R[Q-\delta, Q]$ and which remain a bounded distance from each other for all time in $\tilde{S}$. Therefore, they bound a flat strip by the Flat Strip Theorem, thus contradicting the assumption that γ_1 and γ_2 are regular.

Now, observe that $\left|\Phi(\gamma) - \int_0^Q \phi(g_u\gamma)\,du\right| \leq \delta\|\phi\|$, because we know the period of γ is at least $Q-\delta$. Consequently, it follows that

$$\Lambda_R(\phi, Q, \delta) \leq e^{\delta\|\phi\|}\Lambda(\phi, Q, \varepsilon)$$

where $\Lambda(\phi, Q, \varepsilon) = \sup\{\sum_{\gamma\in E} e^{\int_0^Q \phi(g_u\gamma)\,du} \mid E \subset GS \text{ is } (T,\varepsilon)\text{-separated}\}$. Now, for ε sufficiently small, there exists a constant C by [CT16, Lemma 4.11] so that $\Lambda(\phi, Q, \varepsilon) \leq Ce^{QP(\phi)}$. □

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