Math 229, Spring 2020
Project: The Precession of Mercury

In this project, you will study a differential equation governing the orbits of planets or other satellites around a star. We use polar coordinates r and ϕ to describe the orbit – $r(\phi)$ is the distance of the planet from the star when it is at angle ϕ. We can equivalently keep track of the orbit using $u(\phi) = 1/r(\phi)$. Using the coordinates u and ϕ, here are two possible models for the orbit:

$$\text{Keplerian model: } \frac{d^2u}{d\phi^2} + u = \frac{M}{L^2}$$

$$\text{Relativistic model: } \frac{d^2u}{d\phi^2} + u = \frac{M}{L^2} + 3Mu^2.$$

M is the mass of the star, and L is the angular momentum of the planet or satellite.

The Keplerian model can be derived using gravity as a central force which behaves like $1/r^2$; the relativistic model is derived using general relativity. (You do not need to do those derivations for this project.) They make similar but slightly different predictions about the orbits of a planet.

In your writeup, please include the following:

1. Find solutions for the Keplerian orbit using the initial condition that $u(0) = 1$ and $u'(0) = 0$. What sort or shape do your solutions describe? Does this make sense with what you know about orbits of planets?

2. Under what conditions on M, L and/or r do you expect the two models will give the closest predictions? Does this make intuitive sense? What do any the parameters in your solutions correspond to?

3. One way to measure the difference in the Keplerian and relativistic solutions is by the *perihelion* of the orbit, that is, the angle at which the orbit comes closest to the star. What does a perihelion correspond to in terms of the solution $u(\phi)$? At what angles do the perihelia occur for your Keplerian solution? Now find solutions for the relativistic

model in any way you like with the same initial conditions. Where do the perihelia occur for this model? Can you detect any difference? For this step, use $L = 10$, $M = 1$ just as an example.

4. You should find that the perihelion changes each revolution in the relativistic model, whereas it does not in the Keplerian model. Use the following data to estimate how much the perihelion of Mercury changes each revolution. Read and report on some history of Mercury's "perihelion advance." Why is Mercury the best planet to use for observing this effect?

$$M_{sun} = 1.48 \times 10^5, \quad L_{Mercury} = 2.36 \times 10^6.$$

(Note: these are different from the values you may find online. The reason is that unusual units are used in general relativity where we generally set the speed of light to 1 in order to simplify formulas.)