Math 229, Spring 2020
Project: Radioactive decay chains

The simple radioactive decay model in which one element changes into another is modeled by an exponential DE:

$$\frac{dN}{dt} = -kN, \quad \text{where } k > 0 \text{ is a constant.}$$

N is the amount of the initial element and k is the decay rate for the decay. k is related to the half-life for the element.

Most radioactive decay in nature happens as part of radioactive decay chains involving many steps.

As a simple model, suppose we have a 2-step decay chain A $\xrightarrow{k_1}$ B $\xrightarrow{k_2}$ C. First A decays into element B and then B decays into C. For example, Bismuth (^{212}Bi) decays into Polonium (^{212}Po) via a beta decay, which then decays into Lead (^{206}Pb) via an alpha decay. The constants k_1 and k_2 are the decay rates for these decays. This can be modeled with a system of first-order DEs for the amounts of A, B and C.

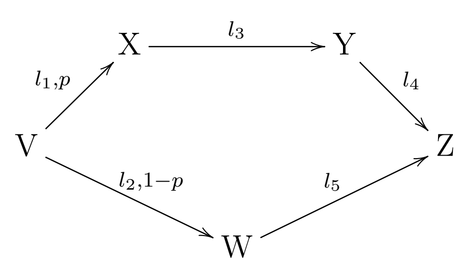
In your writeup, please include the following:

1. For the two-step decay, write and solve a system of 3 first-order DEs.

2. Suppose that you start with 100 moles of A. How long does it take until you have 50 moles of A? 50 moles of C? How long again until 25 moles of A? 12.5 moles of A?

3. Based on your answer to the previous problem, is there an 'effective half-life' for the chain as a whole? Are there different ways to define half-life in this case?

4. Some decay series go through multiple paths. For example:



Here each atom of V decays to Z either along the path V $\to$ X $\to$ Y $\to$ Z, or V $\to$ W $\to$ Z; the first path happens with probability p and the second with probability $1 - p$. Write and solve a linear system for this decay chain.

5. Apply your model to the Thorium series, which you can find detailed here:

   en.wikipedia.org/wiki/Decay_chain#Thorium_series

   Write out the linear system, and use a computer to find solutions for some simple initial conditions. Using your previous work, what should the 'effective half-life' for this chain be?

6. Further up on that page, Wikipedia claims: "When equilibrium is achieved, a granddaughter isotope is present in direct proportion to its half-life". Does this hold for your models? If not, can you find a way to make sense of this statement?

7. If you are working with Thorium in the lab, it is never just Thorium, because it is always decaying. Does your work tell you anything about whether you need to account for the other elements in this decay chain?