Math 229, Spring 2020
Project: The Robertson-Walker Model

In this project, you will study a differential equation governing the large-scale structure of space-time. The Robertson-Walker model describes the evolution of all of space-time and arises from doing some general relativity and solving Einstein's equation – a *partial* differential equation. This is well beyond the scope of this class. Out of this PDE, however, comes a crucial system of ODEs which we can study:

$$\begin{cases} \frac{3f''}{f} = -4\pi(\rho + 3p(t)) \\ (\frac{f'}{f})^2 + \frac{k}{f^2} = \frac{8\pi\rho}{3} \\ \rho' = -3(\rho + p(t))\frac{f'}{f} \end{cases}$$

Here:

- f is a function which describes the scale of the universe at a given time. It is always ≥ 0.
- ρ is a function which describes the average mass density in the universe at a given time. You would calculate ρ by adding up all the mass in a large chunk of the universe (galaxies, black holes etc), and then dividing by the total size of that chunk. (Astronomers have carefully done this.) It is always ≥ 0.
- $p(t)$ is a function which describes the average radiation pressure in the universe at a given time. You would calculate this by recording all the radiation observed at a given time – all the light, radio waves, gamma radiation etc that your telescope would detect. Astronomers have also carefully done this.) It is generally negative. (A negative pressure is an outward pressure in this model – the idea is that the radiation bouncing around the universe tends to push things away from each other.)
- k is a constant which is either +1, 0, or -1.

Our friends in Astronomy have determined some initial conditions for us:

- t_0 represents the current time.

- $\frac{f'}{f}(t_0)$, the current rate of expansion of the universe relative to its scale, is the *Hubble constant*, which is roughly $\frac{1}{18 \pm 2 \times 10^9 yr}$ (you can find more recent measurements online).
- $\rho(t_0)$, the current mass density, is between 1×10^{-31} and $5 \times 10^{-29} g/cm^3$.
- As observed, $|p|$ is currently much, much smaller than ρ. (I.e., there is a lot more observable mass than observable radiation in the universe)

In your writeup, please include the following:

1. Explain why this is a non-linear system of DEs. Perform an equilibrium analysis for any equilibria.
2. Give a short qualitative description of $f(t)$ based on this system. Using this system, prove that there is some time $t_* < t_0$ (i.e. some time before the present) when $f(t_*) = 0$. What would that time represent?
3. Since $|p|$ is observed to be much much smaller than ρ, one simplification of this model is to consider the *dust case*. In this case we ignore radiation, set $p(t) = 0$, and treat the universe as consisting of mass only. Prove that if $p = 0$ for all t then the quantity ρf^3 is a positive constant. Use this to prove that f satisfies the DE

$$(f')^2 + k = A/f$$

   for some positive constant A.
4. Solve the equation in the previous step any way you like for each of the possible values of k. Comment on some features of the solutions. What do they say about the history and future of the universe? Pay particular attention to what happens when f is equal to zero.
5. In 1998, detailed observations of very distant supernovae revealed that the rate of expansion of the universe is increasing. (This discovery was awarded the Nobel Prize in Physics in 2011; one of the recipients, Adam Riess, gave a public lecture on this topic at Wesleyan a few years ago.) Does this agree with your work above? If not, what could be going on to explain it? Do some research on the usual explanation for the acceleration and see how our DEs play a role in understanding it.